Mathematising Mathematics

MATHEMATISED THINKING ≡ META-INTELLIGENT

CATALYSING ECONOMIC MIRACLE

RAMJEE PRASAD
SANDEEP SRIVASTAVA

Mathematatising

MATHEMATICS

MATHEMATISED THINKING ≡ META-INTELLIGENT

RAMJEE PRASAD
SANDEEP SRIVASTAVA

Mathematising MATHEMATICS

MATHEMATISED THINKING ≡ META-INTELLIGENT

CATALYSING ECONOMIC MIRACLE

SANDEEP SRIVASTAVA

Mathematising Mathematics

Copyright© Ramjee Prasad, Sandeep Srivastava, 2023

The moral right of the author has been asserted.

All rights reserved.

No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, without the prior permission in writing of the publisher or be otherwise circulated in any form of binding or cover other than that in which it is published and without a similar condition, including this condition being imposed on the subsequent purchaser.

ISBN: 978-81-959203-6-5

Domain editor: Saloni Srivastava

Editor: Nikita Todarwal

Publishing team: Dipti Chauhan (Lead), Manish Kumar, Tanu Gaur

In partnership with

SANDEEP SRIVASTAVA

www.sandeepsrivastava.online

A-43, II Floor, Zamrudpur, G.K 1
New Delhi 110048
India

For my grandchildren –
Sneha, Ruchika, Akash, Arya, Ayush and Shreya –
Who inspire me to look ahead with joy and zest.

Ramjee Prasad

To the truly educated – reader, gracious, assertive, humanist,
and proud citizen of the society.

Sandeep Srivastava

Dear Readers,

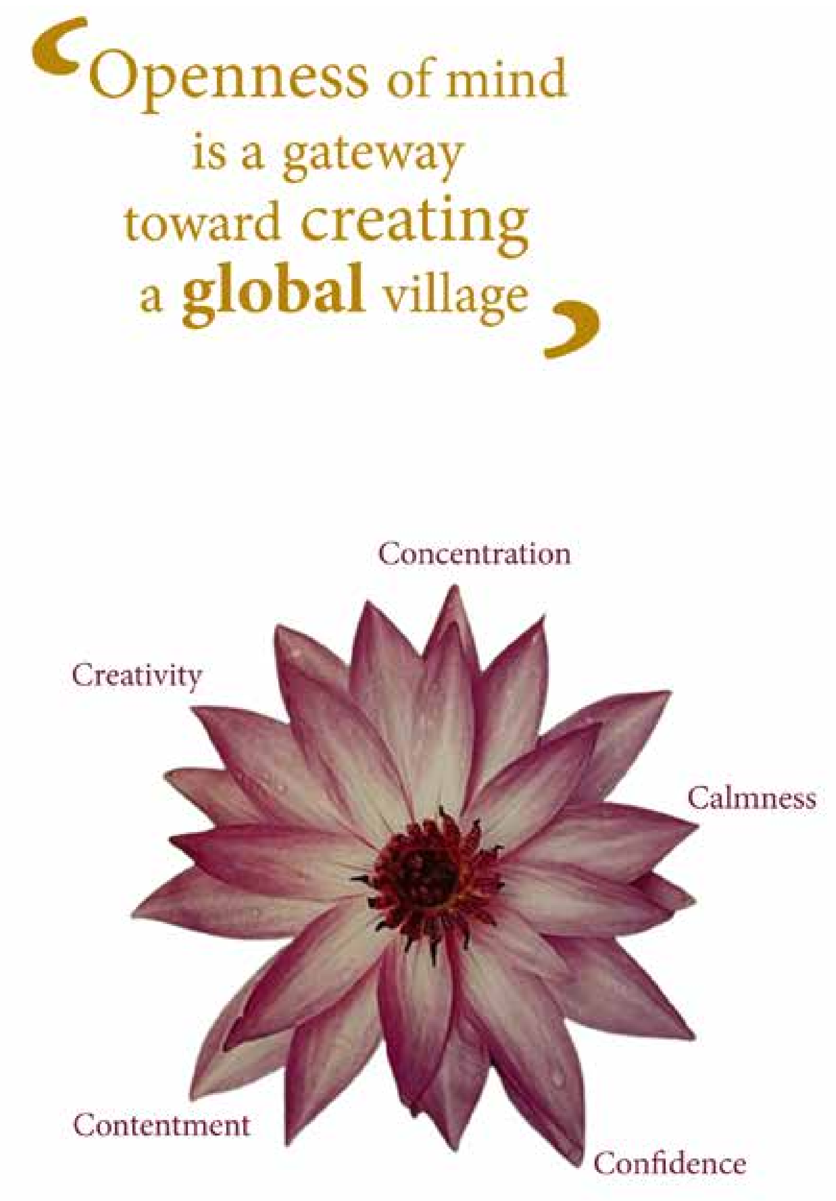
You must know that this book is in a genre of its own – an academically rigorous book for us all globally. For far too long, academic books have targeted only a fraction of the population; even scholastic achievement is taken to be the preserve of the gifted ones.

The tone, organisation of content, conceptual intensity, and expanse of the book qualify it to be an academic work. However, the language is akin to literary writing, and the presentation is 'textbookish' to facilitate an inalienable grip over the flow and substance.

Yet, in the jest to enliven academic concepts, their taut boundaries may have been infiltrated.

Expectedly, the book would not be a familiar feel for all readers; it straddles across the unbridged academic and 'trade' (general public) genres.

We wish hope this inventive format of the book is appreciated.



– Ramjee Prasad
Unlock Your Personalization
Aalborg University Press (March 1, 2012)

This book is also one sequel to 'Unlock Your Personalization.' One end of this book is 'Mathematising (our) thinking' to root mathematics as a language of all social institutions and processes.

'Unlock Your Personalization' promotes an innovative and novel approach to achieving a good quality of life.

Life is short, and its limits are apparent. Living should bring happiness and pleasure, but most people have to cope with enormous problems stemming from heavy workloads, stress, and anxiety. In our post-modern, techno-science world, every effort is being made to achieve a high standard of living. Still, few people find an effective solution for relieving stress and achieving their objectives in life.

– Ramjee Prasad

Content

Preface

Thinking Humankind, All

Swami Vivekananda prophetically called upon us to 'make men first'. He reminded us of the on-the-ground truth that even if governments give us all we must have, *'where are the men who are able to keep up the things demanded.'* Unsurprisingly, the italicised phrase encapsulates the most crippling crisis before humankind a century after its articulation by him. The phrase quite sits at the heart of what the book is set to catalyse – (human) development revolution – by galvanising us all to be spirited humans in the ever-intensifying Sci-Tech (Science and Technology) and AI (Artificial Intelligence) era.

More simply stated, this book is about you, your family, and the community you are nested in. It is most peerlessly so along multiple dimensions. The book is about contemporising you – your knowledge and skills – to the technology of the times – Artificial Intelligence (AI). The book will seed in you the exemplary life to thrive in the fast-realising world of inorganic intelligence, an unprecedentedly malleable future. The book will lay bare the designs and destinations on the Fourth Industrial Revolution (4IR), or Industry 4.0 highway.

The book also decodes why 4IR is struggling to hold its ground, let alone accomplish its promise, and how the context of your family and community cannot be future-proofed if 4IR fails in revolutionising social infrastructure – education and health for all 8 billion of us, and the (economic) dignity of all adults.

The only business of societies and humankind is same-outcome education of all children@18. Health is on the cusp of education and economics. 4IR is our last opportunity to trigger and sustain an unexceptionally global socio-economic renaissance.

Educated blindness – A world hiding in plain sight

Inattentional blindness is a biological limitation, our 'brain's fault'; we cannot really blame ourselves. It is personal in nature, and there is not even a remote chance that two random people could have overlapping inattention to similar things over different times and spaces. On the whole, it all does even out, and no loss or hurt visits anyone; for example, things that get the attention of the sexes are complementary to some extent, and together, a couple covers up for one another (one may miss blue tea roses, the other violet tea roses in a walk through a garden). The worst manifestation of inattention is mostly embarrassment.

There is a social correspond of inattentional blindness, a creation of our socio-cultural conditioning, our 'education's fault'. Unsurprisingly, it is best addressed as 'educated blindness', most acutely and almost universally noticeable among those formally educated, and the longer the formal education, the more likely is the 'affliction' with educated blindness. The only apparent similarity 'educated blindness' bears with its biological twin is that we cannot really blame the educated individuals; the cultural as well as the formal education system is riddled with holes.

Educated blindness is visible and veritable conditioning of thinking, learning, observation, acting to preserve (more significant, long term) self-interests and assertion of moral being. Its worst manifestation is thin-cultured adults, globally with an iron-curtained worldview, a rather narrow worldview, and increasingly transmitted and transplanted worldwide with the help of the most uniform social institution across the world, the K-12 formal education system.

More specifically, educated blindness is reflected in the substantive majority and the 'toppers' of the formal education system falling into a new 'average' – individuals out to pursue similar kinds of careers, having the same meaning of success in career (and not seeking professional stature), same meaning of being rich, the same way of becoming rich, marginalised larger socio-cultural identity, a culture unto themselves and so on. It is the reason behind indifference to unconscionable inequity in wealth and income, fractured society, treating climate mitigation as a new gold rush rather than a social and humanitarian challenge, 'professionalisation' of the social sector – education, health and civil society, and many things else.

Educated citizenship – The vital soft-infrastructure

History is sharp about each new economic revolution stepping up the demands on humans. The era of the driverless cars on the street and the generative AI of the next decade would still need (appropriately) educated humanity. For instance, people 'educated to be doing something joyous' while being driven around, people educated to reinvent their businesses around driverless vehicles, people educated to feel productive and wanted in new ways, people educated to add value to the outputs of generative AI, and more would be in demand. We have to recontextualise what it means to be educated.

However, it seems making humans (raising adults out of infants) is now a long-lost art for humanity. The more 'developed and advanced' the nation, the more vigorous and devoted the institutionalisation of this 'phase of raising, educating' children. Education is the name of all that happens to make a 'dignified, cultured adult' out of infants. Human infants do not grow into humans on biological DNA (all other animal infants do); it takes (more than) a village to raise every child over two decades. Education is 100% social – among role-model adults, peers (ideally not same-age), and the 'real world' (nature and community).

The first, primary goal of nurturing educated adults is to ensure their economic dignity, without exception. The secondary, but not lesser, goal is thickly/deeply cultured adults who live and enrich their chosen socio-cultural contexts, such as active civic constitutionalism.

4IR – The omnipotent hard-infrastructure

To avoid getting the wrong end of the stick, let us begin by emphasising that 4IR represents an entire array of digitised technologies, irrespective of how it is formally defined across institutions and experts. The digital foundation of 4IR (including Big Data), together with AI, is materialising a sea of appropriate technologies to best empower our individual choices. Many of these technological imperatives would require the frontier of science to be expanded like never before and also multiply the intensity of the mathematisation of science (and social sciences, and everything else too). Fortunately, research and innovation processes and resources are transforming to seed unprecedented development of science and technology. 4IR is a dream infrastructure for a socio-economic revolution for all, a first for humankind.

Society 5.0 – Social outcome of 4IR

Society 5.0 is defined as a human-centred society that balances economic advancement with the resolution of social problems by a system that seamlessly integrates cyberspace and physical space. The term “Society 5.0” was introduced by the Japanese government as part of their “Fifth Science and Technology Basic Plan” in 2016 to refer to society that evolves with 4IR. The plan presents the hunting society (Society 1.0), agricultural society (Society 2.0), industrial society (Society 3.0), and information age society, 4IR (Society 4.0). Much like the way 4IR represented the entire technoscape of the times ahead, the gamut of the socio-cultural world associated with 4IR is represented by Society 5.0.

A bit of economic history should help us understand the genesis of Society 5.0 and how a more organic collocation of social and economic development is non-negotiable. The Iron Age triggered a sense of private ownership of anything (land, forests in those times); iron swords, for example, could cut trees to create fields, and animal-drawn carts with iron-rimmed wheels could go way beyond the communal land to claim private ownership. Later, iron weapons made larger-scale war and killing possible, and the sense of private property extended to taking away what belonged to others. Unlike the weapons in the Iron Age, Bronze Age weapons were heavier and did not have strength or sharpness or iron weapons. As pertinently, the driver of 'privatising properties' was the appropriation of extra (societal) resources and later deployment of the differential resources for gaining property.

The race to private property has not abetted even after 3000 years. Worst, the ownership of immeasurable tracts of land (and building as a proxy for space similar to land) is the hallmark of being rich.

Almost as a default implication of private property, economic development, and the good of the entire society have never reconciled to date.

Economic development has come to mean a varying degree, exploitation of communal/societal resources and trust for 'privatising the same', to benefit a few at the cost of the entire society (to which the few may belong). Resource differential still rules the roost, and for all the time in history, it has almost always been the monetary capital. Interestingly, around the cusp of 3IR and 4IR, broadly between the mid-1990s and early 2010s, knowledge was expected to be an equal resource differential, but it is almost back to the historical reality of monetary capital again.

Another (very) long story short, it is rightful to hope that in the 4IR Age, the critical resource differential shall be individuals – their ingenuity, industry, and integrity. In other words, the education system is in its broadest (true) sense. The entire socio-cultural and economic ecosystem is now almost equally accessible to all through the Personal Multimedia Communication revolution. In effect, let us hope and work towards the differential being what individuals can imagine of 4IR and value creation through it for self, family, community, as well as humanity (remember, it is a global village now.)

Interestingly, the website of the Office of the Cabinet, Government of Japan, mentions that Japan aims to be the first state to achieve a human-centred national society, named Society 5.0 (following a particular hierarchy of the evolution of humankind). Such a society could be visualised as one that will create and nurture a socio-economic environment that best facilitates one and all to enjoy a high quality of life, exemplified by proactive and productive citizens, every one of them. It elaborates that innovative culture, diverse technologies, and the social integration of the two into the fabric of the nation are how Society 5.0 will crystallise.

Indeed, over to 4IR and popular imagination.

Section I

AI and 4IR – An inexplicable crossroads for humankind

'Unite to Regulate (Generative) AI' is the new battle hymn of the global political rhetoric, unseating climate change in just a matter of months. 4IR is defying its DNA even after a decade in action – it is turning (net) producers to (net) consumers, crushing economic dignity to dust for an ever-increasing fraction of humankind.

AI and 4IR are inherently about comprehensively personalising the world for every one of us, to empower every one of us to be best educated, healthy, and live with dignity in a truly democratic society (and nation). The two embody the grandest designs for an assuredly happy earth.

Instead, we are staring at an unimaginable and exceptionally collective future – adults, families, communities, and societies disabled from powering their survival and growth. The most gratifying thread throughout history is that families and communities made ends meet, war or peace. All (able) adults were net producers and paid some form of taxes to the state. It is unnatural to humanity to even think of something like universal basic income to a fast-growing fraction of net consumer adults.

Of course, the book is all about negating this possibility. Mathematised thinking is humanity's gravest miss. We present a peerless action plan to mathematise our thinking. To enable every one of us to plug into the truly global and omnipotent AI soft infrastructure and the next-generation 4IR infrastructure. Satya Nadella emphasises the need for a billion developers, considering it a democratising tool to facilitate easier access to new technology and knowledge, simplifying the learning curve. Lest 'a billion developers' is misread to mean 'a billion software/AI professionals,' we wish to assert that it is best read as 'a billion thinking professionals.'

A peerless human revolution is ahead of us!

Fortuitously, the mathematical rooting of this revolution guarantees its infallibility. Mathematics may just be the most explicit self-organising consciousness in us. It is anchored in the ways and working of the world around us as it gathers mass and momentum almost autonomously, on its own devices. The mathematisation of humankind is unstoppable in its reach and catalytic drivers.

4IR Must Ride on Popular Imagination

Thought experiments by a few billion us

A distinguishing feature of disruptive technologies is that they are global; those charging the Fourth Industrial Revolution, 4IR, are no exception. All disruptive technologies are born in the imaginations of a few people; they are hardly evolutionary or get thrown up from the ground realities. It takes someone to transcend the extant technologies, limitations, (insatiable) wants, and economics. Interestingly, the relationship of technologies with imagination is 'symbiotic'; all technologies – local, regional, national, or global – are born in someone's head. Disruptive technologies stand out for the meticulousness and grandness of their imagined avatar, and they are truly elaborate thought experiments. They are explicitly and entirely mathematical in their expression for wider dissemination and operationalisation. It is not wrong to infer herefrom that all technologies are someone's imagination and mathematically expressible (as continuous or discreet mathematical relationship).

Every technology is (somebody's) a figment of imagination

Every 'working technology' is a mathematical success, and failed technologies are, above all, a mathematical lapse. Every technology, good and bad, is first a 'mathematised imagination', starting with a substantial period of subconscious, piecemeal, and gross frames of states and relationships. The self-driving cars, generative artificial intelligence (AI) products like ChatGPT, the Titanic submersible, TikTok interface and algorithms, no-touch water taps, memory pillows, and all else originated as an idea in the minds of some people.

It is no overstatement that the recognition of a technology, the moment the technology comes alive in our consciousness, is filled with mathematical details – polygons, circles, cones, cylinders, sizes, strength, composition of material, price, angles, length, breath, rate of change, etc. Engineering drawings encode the visual reality of the creativity of one or more heads. The prototypes that follow are the beginning of the reality check, yet they have to conform to the evolving images in some heads. For example, every home is first housed in the owner's imagination for an extended period before it is shared with the architects with details of cost, space, rooms, floors, roofs, etc. There is nothing to get started with unless and until there is imagination.

It should not surprise that similar mathematical imaginations operate among scientists and dictate the articulation of the hypotheses to be tested and experimental set-up for the same. But there is a catch here – some scientific 'imaginations' cannot be actualised into an experimental setup. For, science studies nature to draw insights about the 'universal truths' (valid all around, and on earth, at the least proven to be valid beyond earth) behind the order and patterns in nature (for instance, rain precipitates under very specific multi-dimensional conditions).

Scientific experiments have to duplicate conditions in nature precisely; at the same time, many natural conditions cannot be created in all their glory and gusto in laboratories. And then some extraordinary minds push the boundary of imagination to host dynamic experiments in their mind. They live the desired experiment in their mind, thoughts, and pronounce their discoveries that may attract some belief from fellow researchers. Thereon, it becomes a waiting game – fine-tuning experimental paraphernalia that embeds the necessary conditions to endorse or deny the pronouncements credibly. To know that a few thought experiments did come to be true after decades is the most glowing ode to human capabilities every one of us possesses.

A brief introduction to the most audacious thought experiments is worth its weight in gold.

The unparalleled thought experiment

The presence of the gravitational waves was one outcome of the thought experiments of Albert Einstein more than a century ago, in 1916. In his general theory of relativity, he had predicted the presence of these waves criss-crossing the universe as ripples in space-time at the speed of light.

After years of internationally coordinated research and data sharing, especially since 2015, when the evidence of low-frequency gravitational waves was registered, towards the end of June 2023, a consortium of hundreds of astronomers and scientists in North America, Europe, China, India and Australia heard the sound of gravitational waves, much like a continuous hum of a noisy restaurant.

The sound of gravitational waves is being hailed as the most confirmatory evidence of the existence of such waves through the limitless universe. The search for the physical evidence of these 'ripples in a taut fabric of space-time' waves has been active for the past few decades. Michael Keith of the European Pulsar Timing Array stated, *"We now know that the universe is awash with gravitational waves."*

The gravitational waves were stated to be low-frequency waves that travel through the universe and interact very weakly with matter; thus, they are mostly unnoticed. The low frequency of the gravitational waves also meant that their detection was beyond the antennas that we could ever build; low frequency implies large wavelength – the wavelength of the gravitational waves varies from a few kilometres to the size of the universe (there is an inverse quantitative relationship between frequency and wavelength of electromagnetic, or 'similar' gravitational waves). To detect and measure gravitational waves, scientists require extremely sensitive instruments that can detect the minute displacements caused by the waves.

To detect these waves, we had to find a way of discovering a proxy wave that is measurable by the largest possible antenna scientists use (diameter of about 500 m). To understand the scale of things, electromagnetic waves of 3 Hz frequency (3 cycles of 'up and down' of wave in one second) would have a wavelength (the length of one cycle) of 1,00,000 km. For capturing the presence of such extra-large wavelengths, we would imagine antennas of the size of the earth would be needed.

To circumvent this problem, an indirect way of detecting gravitational waves was devised by the scientist. The proxy they used was studying disturbances and small changes in the waves emitted by pulsars (a kind of star that is a source of pulsating electromagnetic waves).

Why do the scientists use the waves emitted by pulsars to detect gravitational waves? Pulsars emit very high frequency waves (in fact, they also include gamma rays, waves with the highest frequency) and thus require a very small diameter antenna to be recognised (given their very small wavelengths), which can be deployed on the earth. When the pulsar waves cross paths with passing gravitational wave, their stable and predictable path and wavelength is altered. Therefore, to detect gravitational waves, scientists track and minutely observe pulsar waves for any change in them due to passing gravitational waves.

In 2015, scientists in the US and Italian observatories announced the direct detection of gravitational waves, which were created as a result of the impact of the collision of two black holes some 1.8 billion light-years away (the event took place so far away from the earth that it took 1.8 billion years for the waves to reach earth, given that it travelled at the speed of 3,00,000 km per second, 9.5 trillion km per year; a distance of 1.7×10^{22}). The collision between the two black holes was so 'energy-intensive, violent' that it sent shock waves throughout space-time (in space, it did not dissipate but stayed alive over time).

Interestingly, the previous three observations of gravitational waves could not confidently fit the rigour of standards to confirm the existence of gravitational waves. It was the fourth such detection of gravitational waves that met the rigorous standards required to verify the existence of gravitational.

It is impossible to detect very low frequency waves by ground-based interferometers or even space-based interferometers. There is no credible source for detecting very high frequency as well. Very low frequency waves can be detected when superimposed on high frequency waves.

This exciting finding proves that Einstein was correct in thinking that the force of gravity is due to the curvature of space and time. It also opens up possibilities of an unprecedented new window onto the cosmos, into the furthest reaches of our universe.

Gravitational waves carry information about their dramatic origins and about the nature of gravity that cannot otherwise be obtained. This collision of two black holes had been predicted but never observed.

Rainer Weiss, professor of physics, emeritus, at MIT, wishfully spoke about how '*it would have been wonderful to watch Einstein's face had we been able to tell him*' about the details of these observations. He eloquently observes that Einstein's theory of general relativity had beautifully described the universal presence of gravitational forces and their impacts some 100 years ago.

Detecting gravitational waves is counted as among the more spectacular achievements in physics so far in this century.

Recall that Newton had universalised the reality of the falling apple in the name of gravity; he had connected the most mundane of natural order (all things always fall back to the ground) to how things happen across the universe. It is not hard to empathise with Newton that he had famously refused to explain the reason for the force of gravity; it was a singular achievement and the first scientific law pronounced to be applicable across the universe.

More than two centuries later, Einstein actually put forward the operational details of gravity. In his vision, gravity caused the universe to go around, be continuously on the move, and along well-defined paths. Gravity also explains why things fall on a body with the same acceleration (the same gravitational force 'area'), their mass irrespective. His explanation of the workings of gravity also accounts for its functioning in a vacuum.

Of course, his work did throw light on why we are stuck to the earth and why things fall to the 'ground'.

Einstein's theory of gravity, his general theory of relativity, has been read to imply that the curved nature of space-time acts as a kind of a (non-material) field of force that acts as a pull towards the heavier object, and in that sense 'things fall towards the ground'. John A. Wheeler, a renowned physicist, famously articulated that the interaction between space-time (points in space at different times) and matter can be described as 'mass gripping space-time, telling it how to curve' and 'space-time gripping mass, telling it how to move.' This means matter and space-time are intertwined and affect each other's behavior. Mass affects how a part of space looks (and behaves); in turn, the affected space navigates any mass around it. It must just help to know that electromagnetic waves, such as light, are examples of non-material 'force', unlike sound waves that are carried by matter.

One of Einstein's most supra-human qualities was his remarkable ability to imagine complex scientific situations as real-life scenarios. He called these scenarios Gedankenexperiments, which is German for thought experiments. He deployed these virtual experiments as his reality check, and it took diverse ways. For example, he visualised what the world would look like if one travelled tagged along a ray of light.

To be honest, what he did was mathematics, as 'armchair' physicist he derived mathematical expressions that represented his imagined reality, and mathematics held for him the immensely powerful logic and reason.

There is a popular and powerful quote attributed to Albert Einstein, commonly known as *'the happiest thought of my life.'* While working as a patent clerk, he suddenly imagined a person freely falling to the ground. His thoughts were arrested around the magnitude of the person's weight, and he could 'see' that such a person would not feel his weight. This realisation profoundly impacted him and inspired him to develop his theory of gravitation. Einstein was so taken aback by this thought that he said, "I was startled."

Thought experiments have a long history

Interestingly, the German literary icon and polymath Johann Wolfgang von Goethe's words on 'thought experiments' resonate with the aforementioned 'happiest thoughts' of Einstein. He discounted 'thoughts', and placed a premium on the 'experiments' with the thoughts. He believed many people have enlightening and innovative ideas, but only those that can withstand scrutiny and become rooted in personal experiences are truly unique and valuable. History is replete with examples of physical and thought experiments that underlie our progressive evolution.

In the seventeenth century, Galileo used thought experiments to affirm his theories. One famous example is that of two balls (one heavy, one light) being dropped from the Leaning Tower of Pisa. And he postulated the behaviour of the falling balls quite opposite to the prevailing expectation that the heavier ball would hit the ground sooner than the lighter one.

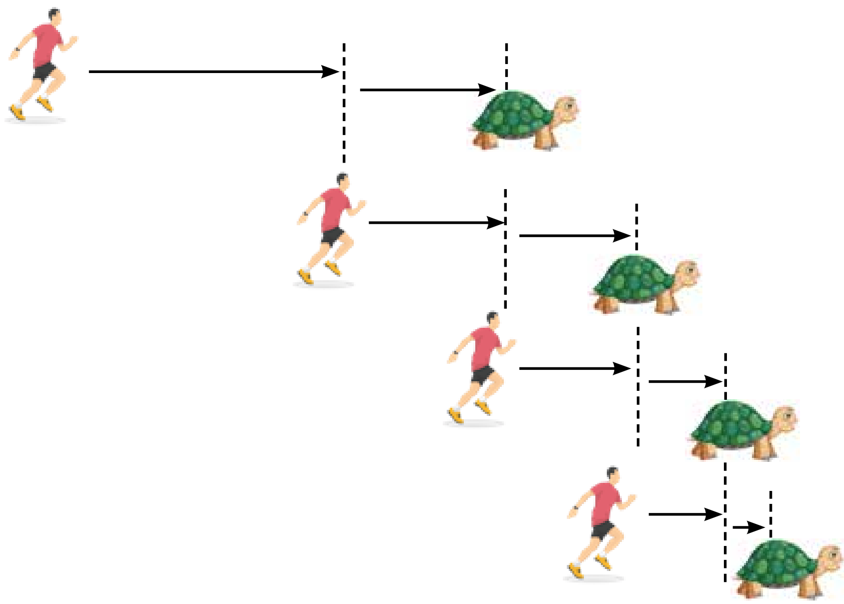
In the 1930s, there was the famous Schrödinger's cat thought experiment. In simple terms, he visualised a cat and something that could kill the cat (a radioactive atom) in a sealed box. One would not know if the cat was dead or alive until the box is opened, and for that duration, the cat can only be assumed to be both 'dead and alive' (uncertainty of death or life.) Pertinently, this state of uncertainty would end as soon as the box is opened; the cat would be dead or alive, not both. The health of the cat could also be seen as 'somewhere between alive and kicking to nearly dead, if not dead'.

Erwin Schrödinger's equation presented a 'similar' (mathematical) description of all possible outcomes, locations and characteristics of quantum systems as they change over time (quantum may be read as 'the smallest quantity', such as an electron or a photon, or even something bigger.) Schrödinger's equation does not specify the exact location of a particle but gives a (definite) probability distribution of the locations of the particle.

Yes, you are correct in thinking that thought experiments must be natural and, in fact, the first step, in setting up physical experiments. The first written record of thought experiments goes back 2500 years to the Achilles paradox. The paradox was the thought proof of the impossibility of infinite distinct actions/steps in finite time. Zeno, a Greek philosopher, visualised infinity as the never-closing gap between a tortoise with a head start with respect to Achilles in a run to a destination, though the gap would narrow with time (depending upon the difference in their steady speeds).

As he saw it, by the time Achilles covers the initial gap with the tortoise, the latter would have moved some distance further ahead, howsoever small, causing a new gap. As Achilles runs to catch up with the new gap, the tortoise establishes another new gap, smaller than the previous gap but a definite/finite one. In the final analysis, there would be a reducing gap between the two but always a (finite) gap, well into infinite, never-ending iterations of the run for the gap. To be true, such a context could only be envisioned.

Visually,



Achilles paradox

The limitlessness of thought experiments

It must be acknowledged that the choice of gravity to illustrate the inalienable role of thought experiments is not just to go behind Einstein's genius. There is more in the public domain on thought experiments on gravity, expectedly, involving another genius.

Richard Feynman and Einstein have a very organic intellectual connection on gravity through a seamless thought experiment on the same (expectedly, both were theoretical or mathematical physicists). This is particularly interesting because Einstein wavered on the existence of gravitational waves; in the 1930s, he wrote a paper denying their existence and later asserted them again in the 1950s. But soon after Einstein died in 1955, Feynman showcased, in 1957, an ingenious thought experiment that sealed the reality of gravitational waves, and all that was left was its experimental proof, which happened in 2015.

The very role of Feynman is humbling and heady. He was known for going back to the first principles to uncover all challenging questions. He was known to be his own person; concepts must concepts must make sense to him personally. And he was a great teacher – his narratives were famously incisive and the simplest. This is primarily why he was invited to the first American conference on general relativity in 1957, despite being at the 'opposite' end of interest in physics – quantum systems. He is highly regarded for 'connecting feeling to doing' and insisting on the high probability of physicalising gut thoughts; in his words, *'my instincts are that if you can feel it, you can make it.'*

Feynman's 'first principle approach' brought him to serve the wake-up call to all – if gravitational waves exist, they must be energy carriers. This concretised the research goal for the waves till their discovery in 2015. His intuitive visualisation of energy has come to be known as the 'sticky bead argument.' The details of this thought experiment are not relevant to our conversations here. Broadly, his visual imagery harnessed the common knowledge of how a pair of anchored (electric) charges oscillated when exposed to electromagnetic waves.

The difficulty in detecting gravitational waves arises because gravity is much weaker electromagnetism; gravitational waves are far less 'forceful' when compared to electromagnetic force. This 'physical insignificance' of the gravitational waves is the major obstacle in their recognition and technological experimentation.

Relevantly, this thought experiment laid the foundation of the effective experimental set-up (detector) for detecting the waves. His 'thought wave detector' helped better mathematisation of the gravitational waves, for instance, to measure the amount of energy gravitational waves may carry. His detector was a simple set up – if multiple detectors are appropriately placed behind one another on the path of the waves, each successive detector would have a decreasing amount of energy registered/absorbed. If this was not the case, then unlimited energy would have been withdrawn from the waves, and that is not a valid situation. Thus, a proof of the waves was in the weakening of the waves when put through detectors.

Six decades later, the waves were scientifically verified.

Thought experiments are better sharable than duplicating experimental setups, making the former a very powerful engine of discovery and inventions.

Mathematics and thought experiments

Gravitational waves as a proof of how Einstein conceptualised gravity came alive as part of the implications of Einstein's equations on general relativity in 1916. However, over the following years, he distanced himself from the existence of the waves, and in a 1936 research paper titled 'Do Gravitational waves exist?' he formally denied it Einstein developed a feeling that the waves were more artifacts of his mathematical presentation of the waves; they did not physically exist. Perhaps, he felt that he exceeded the turf of the geometrical formulation of his space-time conception and its implications. He had based his theory on geometries that were way beyond the established geometry of plane surfaces. He relied on the emerging geometrical formulations of curves and non-planar shapes, specifically Riemannian geometry, which was introduced just a few decades before his theories in 1854. He had drawn so completely from Riemann's work that it would not be wrong to see the mathematical conception of Einstein as applying Riemann in the context of explaining gravity.

A simple fact is that the four-dimensional space-time originated in mathematics, and it was easier for theoretical physicists to visualise, as compared to the larger scientific community. Unsurprisingly, while Einstein revolutionised the understanding of two least understood universal physical realities – energy and gravity, his theories were not instant hits; his language was mathematics, and the mathematics he employed was not so familiar to many physicists of the time.

In hindsight, Einstein's self-doubt is better appreciable, and the words of Steven Weinberg, a leading theoretical physicist and Nobel laureate, put it all in perspective.

According to him, by the 1910s, the decade that saw the articulation of the theory of general relativity, Riemannian geometry was gaining wide acceptance in explaining the behaviour of smoother curves (best suited for modelling the way the gravitational field is visualised as taut fabric). Recall that the more the curvature at a point, the greater the gravitational force experienced at that point; understanding the nature of the curvature is all important. But it must also be noted here that he does accord all admiration to Einstein for visualising gravity to be not a force, per se, but the effect of '*the peculiar empirical properties of gravitation (of curvature, and shapes that curvatures make).*'

Einstein's extensive thought experiments materialised on the back of rigorous mathematical reasoning, high integrity in the application of established mathematical formulations, and meticulous computational details.

Indeed, mathematics gives thought experiments the much to be 'seen', repeatable, communicated, debated, and experimentally verified.

The essence of thought experiments

Let there be no misconception of thought experiments being critical to sciences and interdisciplinary domains such as urban planning also deploy (mathematical) visualisations. For example, no on-the-ground study can be initiated to green up cities through non-fossil energy as the current and the eventual urban spaces look vastly different. The design and development of densely populated cities follow their energy source – high energy-density fossil fuels. Cities would need to be 'refounded' for the much dispersed green energy; solar energy, as per the current solar cells, takes a few hundred times more space to produce energy than fossil sources. This transformation of cities has to be a thought experiment, an imagined new model of cities to start with; the scope and size of such a change are too massive to be concretised by any formal knowledge and tools.

Thought experiments are also used in social sciences and even philosophy. A common trigger for thought experiments in these domains is the exploration of social conditions that involve ethical, moral, and increasingly religious and political dimensions, which are also best thought experimented on. Top of the recall examples of such ethical issues would include those related to mercy killing, abortion, affirmative actions, gender identities, and even children's rights, which are hotly debated.

In broadest terms, we may dilute the reference to mathematics as a foundational vehicle for thought experiments, but they remain fairly well-defined and well-articulated. Richard Feynman calls thought experiments to be scientific – *'a process of thought based on integrating previous knowledge, observing, measuring, and logical reasoning.'* Yet, it is much about the person who lends the head for it.

It is no exaggeration that thought experiments empower us to investigate situations that are impossible to test and to predict their behaviour confidently. Mastering thought experiments may be the only way to confront the most challenging and intriguing situations and anticipate (and prevent) potential crises.

Thought experimenting 4IR to infinitise 4IR

Why thought experiment 4IR? Whatever 4IR stands for in popular imagination and the scientific community, business organisations, and governments, it has failed and means little to the vast majority of humankind. We have to reimagine 4IR.

Also, let the association with 4IR not imply any limitation to the technological juggernaut of these times and its acceleration in the decades ahead. 4IR is just the most popular symbol of unprecedented possibilities.

We have already alluded to 4IR being a momentous opportunity. time, the 'world of one' is here and now; the world is the stage for all, everything is accessible to all for just deserving them. Personalisation of education, resources, and opportunities is increasingly without bounds.

Mass personalisation is passé, for all the good, it still leaves much to be desired. The intensive digitalisation of the most micro of processes and resources in value chains of service and production implies dream combinations of pathways to create personalised, best-value experiences and products. This is almost like creating value out of thin air, maximising value added at every discrete step, addressing every bit of unfulfilled want/demand while completely erasing 'consumer waste' (kind of extra, unsought, less-than-useful features).

Next-generation personalisation may best be named 'Living Personalisation' (LP) – creating products and services that dynamically and continuously improve towards zero 'consumer waste', and no unfulfilled demand. It rides on Big Data on individuals, their dynamic choices, and longitudinal generative AI. Personal Multimedia Communication (PMC) is just the infrastructure to make LP live and kicking. 4IR is just the comprehensive ecosystem around that infrastructure, encompassing social, cultural, personal, economic, political, technological, and ecological dimensions.

People must be able to generate new and more cost-effective socio-economic values out of the infrastructure. However, benefitting from 'personal infrastructure' is a different ball game, a unique and evolving opportunity, compared to making the most of mass/public infrastructure. Additionally, personal infrastructure is an equalizer like nothing else before; it evaporates the access barriers, and in that process, it also dramatically intensifies the competitive landscape despite accentuating more personal/private competitive advantages.

PMC's promises are only realisable when the users grow to be naturally technology savvy, have a fairly practised scientific temper, have a mathematical sense to express relationships among quantities and have critical and creative thinking, as well as specific mathematical competencies that help in Big Data analysis (patterns in real Big Data are emerging as a new source of research and innovation).

Clearly, 4IR needs to be (continuously) revisited and experimented with again (and again) to realise its potential promises.

What may be thought experimenting 4IR? To the extent that 4IR is an ecosystem, it is not 'one thing, or one situation', not even many things or many situations. How does one conceptualise such a boundaryless 'entity'? We cannot. Fortunately, two dimensions are foundational to 4IR – technologies and people (how we interact and react to the unforeseeable deluge of new sci-tech and how we collect together to live the most sci-tech of times).

To the point, substantively, thought experimenting 4IR would be about re-imagining the course of science and technology that would maximise the well-being of all of us (not the maximum possible number of us). Studying technologies and 'who we are, what makes us so' may be the primary focus. Mathematics would be a big part of the technology rethink.

As a process, it should integrate all the extant knowledge, newer observations and experiences, logical reasoning around technology, human thinking, values, and behaviour.

'Infinitising 4IR' is the personalisation of the world

At the ground level, 'infinitising 4IR' means leaving people to make sense of the 4IR ecosystem in their distinctive, differentiated, and dynamic ways and respecting individuals to maximise their existential and meta-physical identities and joys. The cusp of emergent personalisation-enhancing technologies and every 8 billion of us is limitlessly diverse and deep.

At the level, 'infinitising 4IR' means ensuring and sustaining organically assertive societal oversight over the choices open to communities, businesses, and governments, in all matters impinging upon civil liberties of people. Effectively, this is about assuredly protecting the aforementioned ground level meaning of 4IR. One critical dimension of 4IR is enabling ever-refined, truly democratic social consensus processes. We cannot even well-describe the imperatives, possibilities, and gains of a 100% inclusive and transparently negotiated social decision-making process. To be true, the revolutionary impact of 4IR as a context for revitalising societies cannot be finitised.

Very interestingly, the catalytic possibilities of 4IR at personal and societal levels also pulls it in fuelling an educational renaissance in nurturing infants into adults who shall play their part in infinitising 4IR, as above. A same-outcome formal education system (K-12 at the least) is needed to ensure adults are ready to make considered choices in the 4IR ecosystem.

Popular imagination will infinitise 4IR

Everything is getting digitally denominated. Digitalisation has grown exponentially, both laterally and vertically, and the two expansions act to be mutually additive. It is democratising innovation, application as well as scientific (real-time, Big Data is powering bottom-up innovation). On the whole, the ecological footprint of digital technologies is also favourable. It catalyses AI, but the idea of AI is not new. It also alters the human-human interface in other ways – virtual and augmented reality. Above all, its disruptive force is fast, furious, and sweeping.

Most importantly, digital technologies continue rapid mutation apace – getting 'softer' by the day as the hardware is getting literally and figuratively miniaturised. In turn, the softwarisation of digital technology is limited only by our ability to logicalise situations and logicalisation is actually mathematisation, our ability to express given situations through mathematical expressions.

By definition, new applications of digital technologies often follow mathematical discoveries. Infinitisation of 4IR is limited only by the mathematisation of (our) thinking – for both the halves of 4IR – technological innovations, as well as people capacity building. Mathematisation of thinking is just the imperative.

One thing is absolutely certain – 4IR will keep expanding in scope and scale, and mathematics will be marching in step, actually ahead.

Humankind is mathematising!

Humankind is Mathematising

Soaring discovery, invention of Order

The place of mathematics in society has never been in question. The pedestal accorded to mathematicians across societies volumes about it. Mathematical knowledge and skill are considered such a suprahuman occurrence that even a poor academic record in mathematics is some kind of a badge of quality of being 'normal', being creative in some ways. People feel cringed at not being good at drawing, singing, or dancing, but being poor at mathematics is a matter to bond over. Apparently, mathematical handicaps seem to bear little correlation with professional success. To top it all, mathematics skills need not be personally mastered because mathematical computations enjoy the benefit of high integrity. Misdemeanours and even frauds never go undetected for long, especially after due forensic examination.

However, all this leniency with lack of success in mathematics education started to pinch as the Third Industrial Revolution, 3IR, peaked towards the last decade of the twentieth century. Chorus for improving math education globally has been getting shriller ever since. 3IR matured with analogue electronics turning to digital electronics; semiconductors, personal computers, internet, mobile phones, robotics, 3D printing, and the like were the typical innovations of this era. 3IR brought down barriers to opportunities and resources to ignite knowledge-led, small-business entrepreneurial ventures to compete with and complement corporates.

The Fourth Industrial Revolution, 4IR, has progressed the techno-economic cart to bring more disruptive, global, and personal force to innovations. 4IR emerging technologies, such as IoT, 5G, Big Data, Self-organising supply chains, Next-generation (DNA) Sequencing, Artificial Intelligence, and Blockchain, are significantly differentiable from the Third Industrial Revolution, 3IR and the Second Industrial Revolution, 2IR. Technologies for being duly more general and original. The more general a technology, the more flourish they bring to technical progress and economic growth. The more original a technology, the more refined it is; using more diverse domains of knowledge in technology is one way to make it more original. Originality and generality trigger newer products/services and newer markets/people, respectively; for instance, generative AI has caught the imagination of employers for surgical excision of their customer service teams, just as some rudimentary AI application is already reaching all 3 billion Internet users.

It must be admitted that these technologies are similar to their 3IR roots. However, their extra thrust lies elsewhere – more science (new and refined) and more rapid innovations (shorter 'Technology Cycle Time'). Technology Cycle Time is broadly the period in which technology lives in its original suit. It involves the invention of technology, its justification as superior, purchase and deployment, sustenance/maintenance, and till the emergence of a better technology. In a way, 4IR is a 'general purpose (sci-tech) ecosystem', so vastly expansive that it will underpin infinite 'plug and play' choices for people.

Soft is finally the hard, new power

The feminisation of humanity is undeniably set. Multiple gender recognition is steadily mainstreaming. Corporates are softening institutional culture to retain one and all, despite customers picking up the cost of the slack (is there ever a free lunch). The notion of soft power still has its believers in the conduct of international relations, especially made possible in the times – of collective climate response, the Internet, global supply chains, migrations, etc. In many sections of societies across nations, soft is becoming quite aspirational, being equated with being wise.

The soft power in the digital age is on another plane. 3IR was hard powered, and 4IR is soft powered – driven by mostly soft – software codes, digitalised data, mathematical models, self-correcting codes, cloud storage, non-resident stored procedure codes, soft reboots, soft memory boxes, and more, and new softer dimensions.

Is there a softer of the soft? Yes, mathematical models! Besides, mathematical objects (numbers to proofs) are the steel of every example of the aforementioned soft dimensions.

Indeed, as 4IR becomes an integral part of all of us, it is no exaggeration that humankind is mathematising.

The shreds of evidence of steadily mathematising humankind are overwhelming.

Technology – The embodied mathematics

Technology is the outcome of applying established scientific knowledge at a given point in time. It manifests as tangible products (such as water in a tap or pen) and intangible services (such as software codes, and Internet search engines). Regardless of form—whether physical (as in a pen) or digital (like Google's search engine)—all technology is governed and controlled by mathematical principles. For instance, a pen: its ink flow, the pressure required on its tip, and the design of its grip, etc. are all governed by mathematical parameters. Any sudden deviation in the established mathematical parameters would make the pen unusable. Likewise, something as intangible as Google's search engine also demonstrates a deep mathematical structure behind its functioning, making it a fascinating subject to explore.

Big Data Turns Simple into Great

Google's search engine needs no introduction; it's arguably one of the most popular technologies in use today. It is well known that the engine employs a simple algorithm for arithmetical count, called PageRank, to assign a 'relative importance' to webpages, determining their rank across millions of websites. For the curious, the PageRank algorithm does not 'see' the contents of web pages for ranking. Instead, it measures their popularity by evaluating factors like how frequently they are visited, their link structure, and how often they are hyperlinked by other pages.

Mathematically, PageRank assigns webpages a score between 0 and 1, representing a probability. A PageRank of 0.5 would imply that there is a 50% chance that a user clicking on a link (for certain keywords) would lead to that webpage. However, the process of computing the PageRank score is more complex than a simple probability computation, The algorithm continues to evolve, leaving us constantly speculating about its exact workings.

PageRank is a specific-patented process for ranking webpages. In addition to this, Google uses other methods, such as the mathematical model known as the Markov chain. A model is simply a quantitative relationship between variables—like the formula for the circumference of a circle in terms of its radius. The Markov chain model is based on probability. It represents a sequence of possible future states or situations, where the transition from one state to another occurs randomly, and each transition has a specific probability associated with it. It is especially useful for predicting changes in systems that evolve over time.

The strength of the Markov chain lies in its simplicity. It predicts the next state or step based solely on the current situation, without needing any information about how the current state was reached. Markov chain mathematises the relationship of the change in such a way which makes it ideal for real-time decision-making, such as finding alternate routes during a traffic jam or choosing optimal communication paths in a network. Remarkably, the accuracy of these predictions, based only on the present state, is nearly as effective as predictions made using knowledge of all past situations.

Public key encryption forms the mathematics foundation of Blockchain technology. One of the most popular applications of Blockchain networks – cryptocurrencies – is so deeply rooted in mathematics that they may be thought of as "mathematical money." Their creation and value sustenance is a mathematician's delight. It involves very complex mathematics and computational methods. Expectedly, it has led to the development of mathematical models with applications beyond cryptocurrencies; the models used for pricing cryptocurrencies are so expansive and exhaustive that they could be used to estimate the 'market (or reasonable) value (price)' of many things.

Real-world is So Much Calculus

The fundamental connection of calculus to everything beyond straight and standard geometry (non-linear and non-standard) prompts that 3D printing technology must be drenched in mathematical soundness. The dynamical properties of objects to be 3D printed, such as the strength of the initial conditions/steps/formations, shapes, stability of the entire print, and the sharpness of the contours, are just some dimensions that must withstand the scrutiny of mathematical modelling. Differential equations are extensively used in the software employed in 3D printing.

Big Data Reading and Management

Technology discussions are incomplete without taking a peek into how TikTok redefined timelines to become one of the top social media applications. Eleanor Cummins, a freelance science journalist and an adjunct professor at New York University, brands

TikTok's algorithm as "all-knowing", and contrasts it with Facebook. She notes, "Whereas Facebook asks you to set up a profile, and hand over a treasure trove of personal information in the process, TikTok simply notices—or it seems to."

TikTok exemplifies how Linear Algebra, a relatively simple branch of mathematics, is rightly recognised as the "mind of machines" (facilitating machine learning). In this context, Eleanor accurately describes how mathematics has "flattened humanity into a series of codes." This invisible hand at work, is akin to a supernatural force, moving and shaking everything we experience in our technological world.

New mathematical modelling is also redefining medical diagnostics. For instance, MRI machine designs are problematic for claustrophobic patients and are noisy for everyone. Speeding up the scanning process has been a top priority in MRI research. Recent mathematical breakthroughs have come to the rescue of patients, doctors, and biomedical researchers. Mathematics made high-fidelity compressed sensing, a reality by compressing patterns of '0' in captured digital images, a technique commonly used to reduce the file size of MP3s and JPEGs. As a result, MRI scans that once took 5 minutes can now be completed in just 30 seconds using data compression models.

Mathematics is becoming an even more integral part of our daily lives.

Science – The divided house

Natural science is the body of knowledge that codifies the universal principles governing both living and non-living aspects of nature, as they operate in, on, and around the Earth. This field is divided between mathematised sciences such as physics and much of chemistry and descriptive sciences, notably a significant portion of biology. Additionally, all biological and medical technologies are, by definition, mathematised; drugs, blood tests, scans, and similar technologies are deterministic when aligned with clinical diagnoses.

Mathematics serves as the sole language of physics, with a thriving community of mathematical physicists dedicated to the mathematical foundations of theoretical physics. Mathematical concepts and objects are the tools for conducting thought experiments in theoretical physics. Any discovery and development in physics are inherently mathematised, whether theoretical or experimental. Viable experimental setups rely on technology, and mathematics. For example, a prominent area of research in physics, quantum mechanics, is already such that a rigorous description of quantum mechanics is purely mathematical.

Natural science is the body of knowledge that codifies the universal secrets of both living and non-living nature, as it operates in, on, and around the Earth. It is evenly divided between mathematised sciences (such as physics and much of chemistry) and descriptive sciences (a significant portion of biology). Additionally, all biological and medical technologies are, by definition, mathematised; drugs, blood tests, scans, and similar technologies are deterministic when aligned with clinical diagnoses.

Mathematics is already the sole language of physics, and a thriving community of mathematical physicists is focused on the mathematical foundations of theoretical physics. Mathematical concepts and objects are the tools for conducting thought experimenting in theoretical physics. Any discovery and development in physics are automatically mathematised, whether theoretical or experimental; viable experimental setups are the work of technology, and mathematics. A prominent area of research in physics, quantum mechanics, is already such that a rigorous description of quantum mechanics is purely mathematical.

The Long and Short of Mathematics and Science

The story of theoretical chemistry mirrors that of physics; mathematics is the medium for its thought experiments. This discipline draws from physics, mathematics, biology, and computing to further investigate and simulate molecular behaviour, develop new molecules, and advance new theories.

Biology, like other sciences, is also gradually adopting the language of mathematics, although it is not as mathematised yet. This is due, in part, to our ongoing need to understand much more about both our commonalities and unique characteristics. Biology examines the physics and chemistry of living beings, while the latter two fields focus primarily on the non-living world. The essence of life remains mysterious, and whether we fully comprehend it as humans or through AI remains to be seen. However, the secular trend is undeniable—biology is becoming mathematised; albeit slowly but steadily.

The Cognitive Mathematisation of Science

Interestingly, the cognitive mathematisation of science began with Galileo, who integrated mathematics and astronomy (and later physics). It is worth noting that what we now call science was known as natural philosophy during his era, a term used until the early nineteenth century. His perspective on mathematics as the language of science remains the last word on it to date. He referred to nature's working metaphorically as the 'book of nature' which could only be read through mathematics.

To Galileo, mathematics was the language of nature; he saw a mathematised nature. He further emphasised that a philosopher must also be a mathematician, and in the process, he separated 'pure mathematics' from mathematics used for understanding the real, physical world. In the early seventeenth century, he recognised that nature's mathematical workings were too complex to be easily grasped and appreciated by most people. This view also acknowledges that mathematics was highly idealised in a world where speed and shape vary infinitely (astronomer as he was).

Mathematics as the Code for Order

In what may be seen as an extended interpretation of Galileo's views on mathematics, he argued that mathematics serves as a means of simplifying the complexity of nature and abstracting what is physically undefinable. Without mathematics, how could we even attempt to paint a picture of the universe in mere "words"? The mathematical language of his time was predominantly geometric, whereas the prose of natural languages was often too verbose to capture nature's complexity. Galileo believed that mathematics was the only way to decode nature's existence, leaving no room for ambiguity or the "play of words." His belief in mathematics was almost metaphysical; he argued that humanity would be much worse off without the mathematical order of the universe, which is for us to decipher. In his words, *"Nature is inexorable and immutable; it never violates the terms of the laws imposed upon her."*

He maintained that natural languages, best suited for scriptures, allow for interpretation and contexts that can be understood by all. In contrast, nature is constant for everyone at all times, but it requires mediation to be "read," and this mediation is accomplished through mathematics. Galileo's conviction and enthusiasm for a mathematised science created a mystique around mathematics that transcended domains of knowledge and sparked a general academic reevaluation. As Hardy Grant elaborates in his book Turning Points in the History of Mathematics, *"The clarity of ideas and the certainty of inference characteristic of mathematical thinking became beacons… providing a model for those who would organize and expound their realms."*

Four centuries later, nature—and therefore humankind—must be far more mathematisable.

The three sciences are steadily being unified through mathematics!

Research – Halo to hello

This post is not the space to talk eloquently about how scientific research is mutating; several tomes around it are already on the shelves. Instead, I would like us to explore instances of new scientific discoveries that are almost entirely driven by mathematics, and how the proportion of purely descriptive scientific knowledge is rapidly shrinking. I believe the future may pose challenges for securing time and money for hypotheses that are non-mathematisable.

Scientific research is advancing at an unprecedented pace due to generative mathematical modeling and ever-expanding computing power—both of which are mathematical marvels. Notably, this research revolution has many faces: "real-world measurements negating the need for traditional experiments and simulations," "continuously learning, generative artificial eyes and minds replacing human observations," "mathematical models capable of identifying patterns in real (Big) data —whether new, old or highly diverse," "more seamless collaboration between scientists and mathematical modeling," and "the degentrification of research, moving away from high costs and reliance on the best minds."

Mathematising Reality – Simplifying Possibilities

The recent achievement of complete genome sequencing, in just over five hours—a Guinness World Record—exemplifies the cutting-edge intersection of research, innovation, and mathematics. It also underscores the growing impact of mathematical models and computational power, which require massive data processing, adding to the climate threats. A new ultra-rapid genome sequencing method developed by Stanford Medicine scientists, and partners, has made it possible to diagnose rare genetic diseases in less than ten hours—a previously unimaginable feat. Professor Euan Ashley in the team reported: –'This diagnosis (took place) in about the time it takes to round out a day at the office'. This stands in stark contrast to the few weeks typically considered "rapid" for genome sequencing by most clinicians.

Genome sequencing reveals a person's complete DNA makeup, offering crucial insights into diseases rooted in genetics. This leads to faster and more precise diagnoses, targeted treatments, and significantly reduced patient costs.

The mathematical brilliance behind this breakthrough lies in the '"long-read" sequencing technique (that is, Whole-read); which reads the entire genome at once. In contrast, the standard genome-sequencing techniques simplify computational and mathematical challenges by slicing the genome and then building it again after analysing the sequence of DNA base pairs in each slice. This whole-read method has about a 12% higher success rate than the average rate for diagnosing hard-to-detect diseases.

Another critical factor in this achievement was the unprecedented speed of data processing. The sheer volume of genomic data had overwhelmed the facility's computational systems. Professor Ashley candidly worded the situation: (we had to) "completely rethink and revamp our data pipelines and storage systems" to efficiently handle and process such massive datasets at the required speed.

To grasp the scale of this data, a single human genome contains 3.2 billion DNA base pairs, each encoded by two bits, resulting in about 800 megabytes of digital data. But, for the sake of uncompromised accuracy, genome sequencing is repeated multiple times, often expanding the data to tens of gigabytes per individual. Processing such large amounts of data takes time. To address this, scientists compressed genome data by focusing on differences from a reference genome sequence, bringing the data down to just a few megabytes, making it much more manageable.

Generative Reality: Virtualising Using Mathematics

Generative reality, driven by mathematics, is revolutionizing drug development. AI is now used to train neural networks to test entirely new molecular structures for their interaction with targeted bacteria or viruses, eliminating the need for physically creating molecules or setting up experimental environments. These AI platforms can screen over a billion molecules to assess their effects on a pathogen, a scale far beyond the capacity of traditional physical methods, which typically test only a fraction of that number.

Reducing Research to Data-peered Reasoning

AI is fundamentally transforming the research process—data now serves as a strong foundation for generating and testing hypotheses. Experimental setups are brought in only as needed, primarily for evidence and to validate specific aspects. Massive longitudinal and latitudinal datasets ("Big Data") allow 'AI software' to detect patterns, if any, and articulate them based on their training. The AI continuously reanalyzes the entire dataset from multiple angles and designs, yielding deeper insights and more robust results.

In fact, AI can now generate hypotheses directly from data without the need for training on predefined models, theories, or possible real-world correlations that the data may represent. All that is required is the high-level computing power and basic statistical tools to begin identifying patterns and generating hypotheses in the context of the data's real-world correspondence.

Chris Anderson, former editor-in-chief of WIRED, in his visionary article, way back in 2008, 'The end of theory: the data deluge makes the scientific method obsolete,' predicted, this shift that in the age of petabyte data and supercomputing, the traditional scientific method could become obsolete, with experimental evidence for hypothesis testing becoming unnecessary and inefficient.

Anderson emphasized that at the petabyte scale, information transcends traditional three- and four-dimensional models, evolving into "dimensionally agnostic" statistics. He predicted that, in the future, the hypothesis-driven scientific method would eventually be just one of many approaches to studying the world.

The key, he argued, lies not understanding why things behave as they do, but tracking and measuring what it is with unprecedented accuracy. The "tracking and measuring " refers to Big Data—the extensive, real-world record of the actual behaviour of the target object(s) or situation(s).

While simulations may still be desirable, they are no longer essential. AI now can dig deeper into data and identify patterns, gaps, anomalies, and dimensions beyond human capabilities. However, to be true and fair, the process of extracting meaningful insights from Big Data requires more than just mathematical prowess and computational strength—it also demands human insights, intuitions, and imaginative rendition.

We Must Get It Right This Time

The evolving reality for the scientific community is that with lots and lots of data —potentially infinite—we could hypothetically recreate all the scientific knowledge to date. Nature's ways are constant, and science is merely our attempt to read and explain it. Yet, there is an insurmountable gap in capturing infinite data from nature. The nature of this infiniteness is exemplified next.

The upcoming Square Kilometre Array (SKA) Telescope located in Australia and South Africa is an excellent example showcasing the quantitative sense of the infinite nature of (one) Big Data that is powering the new hope and promise. SKA is an intergovernmental project designed to explore and take a peek into deep space to gather data to understand better some of science's most complex questions and humankind's oldest mysteries, including the potential existence of intelligent life elsewhere in the universe. An analogy to illustrate the vastness of Big Data consider this: the SKA will generate an amount of data in just one day that is equivalent to the entire planet's data output in a year!

And there are infinite such 'Big Data' that we need ahead!

There is More Than Big Data Vacuum

Is there a spanner in the works for the 'Big Data promise'?

Yes, there is – Mathematics.

The tools used to analyse Big Data often have elaborate assumptions and limitations that are not always easy to register and navigate even for mathematicians and scientists. All mathematical models rely on specific assumptions about the data, which may not hold in real-world scenarios (for example, assumptions of independence or linearity between data points). Mathematical representations often simplify complex, real-world phenomena, which can result in 'overfitting or oversimplification' of contexts within the mathematical framework.

Moreover, there is a tendency to rely too heavily on these models, turning them into "black boxes" that obscure our ability to critically evaluate the conclusions or predictions they generate.

So, while Big Data holds immense promise, challenges remain—particularly in terms of the limitations of mathematics and the difficulty in securing hi-fi, comprehensive, and complete data necessary for its full potential to be realized.

Anthropocene = Research avalanche

All this brings us to a growing impulse to recharacterise the Anthropocene era around research (and innovation) – it is a peerless capability enhancement for individuals, communities, and humanity as a whole, positioning research as a remarkable source of positive impact on our lives, Earth, and the environment. With the unprecedented democratization of research, in the times to come, individuals and communities in the future will be far better informed and more empowered to assert the most optimal solutions.

The power of diversity, untrained expectations and goals, lived and unique experiences, unfettered wants, and needs, and the imagined world we wish to create and dwell in, will all become sources of valuable and 'good and grand' research.

This shall also include the salutary effect of liberating scientists from the constraints of pre-research grants and permissions. And this change is already on the horizon. For instance, as Eric Schmidt describes, there are ‘self-driving labs’—AI-powered virtual research labs that people can hire.

And what is the potential of AI-picked research? All this does herald the rise of AI as research sovereign, with AI analytics suggesting and accepting research agendas for the future.

In short, all of this forms the perfect substrate for recommitting ourselves to the goal of harnessing the explosion of scientific knowledge for the benefit of all of humankind, all!

Research is following its master – as the theoretical sciences expand, research increasingly mathematises.

Innovation – The New Socio-Economic Sprint

What research is to science, innovation is to technology! And there is a data-led radical shift in the process, value-addition, and mathematisation of innovation. We already live in a world where intensely Big Data-driven companies are likely to be several times larger in top line, bottom line, and customer service compared to traditional technology companies.

We will now explore some well-known examples of innovations driven on the back of Big Data.

Google – Data-driven Business Design

Google is the world’s most ‘data-denominated’ business and the largest user of Big Data technologies. It showcases all that is possible with digital, mathematised, Big Data innovations – high-performance organisational design and culture, rapid product redefinition, evolving revenue models, massive post-sales support, and a formidable product pipeline. Yet, all these do not imply that Google gets it all right and thriving. There is much more to valuable innovations than just scale.

Google's pioneering work in advertisement-backed business models is essentially applied mathematics. It bets on generating large and varied datasets from user behaviour, continuously developing and tweaking analytical models to mine the user data, and optimising product features to service a large population at manageable costs. To best sustain this organisational DNA, Google has also become 'employee-centred' in several groundbreaking ways - many of which are part of public history.

Google Translate – Data-powered Thinking

Google's Universal Translator, unveiled in 2023, is truly a technological marvel. It can seamlessly live-translate video content across 300 languages, syncing the lip movements in the original video for each translated language. By 2023, Google had amassed 17 years of longitudinal data on nuanced translations made by native speakers. The importance of Big Data is clear, especially in how the Translator works more efficiently for "high-resource languages" such as German, where there is a wealth of written material to train the AI.

However, translation performance in "low-resource languages" (those with limited availability of literary and communicative reference texts) tends to be less satisfactory. Despite this, improved algorithms continue to enhance performance. In the end, Big Data plays a decisive role in the quality and scope of AI applications.

Big Data: Both an End and Many New Beginnings

Peter Norvig, a distinguished researcher in human-centered AI, aptly sums up the role of algorithms (software-based mathematical models) with his famous quote: "Essentially, all models are wrong, but some are useful." If this seems abstract, Chris Anderson's perspective clarifies the importance of Norvig's approach. He highlights the rapidly emerging reality where computers rapidly learn models from data, bypassing the need for humans to derive models through extended contemplation. In the end, all knowledge should be thoroughly logicalisable and mathematically modelable. However, we still lack the mathematical models to interpret many types of Big Data, and we do not have the necessary data to drive data-powered research and innovation in most fields.

Not surprisingly, Google Translate improves its algorithms by generating 'synthetic parallel data'—essentially creating data to fill gaps. The system leverages lessons from same-family language translations, multilingual translation, better identification of 'noise' or missing genres in training data, and more refined integration of different dialects. The remarkable aspect of these algorithms is their ability to 'think in language.' They are highly trainable. Given sufficient Big Data, Google Translate could translate Hindi to Farsi as confidently as it would translate Arabic to English. Of course, this "thinking" is rooted in mathematical logic, a testament to the intrinsic connection between mathematics and languages.
Amazon, Netflix, Uber, Coca-Cola, McDonald's, Zomato, Starbucks, and MasterCard are just a few companies following this data-driven path pioneered by Google.

But What Is Big Data?

The Jack Ma-founded Ant Group in China offers one of the best examples of Big Data's potential, closely mirroring Google's data-driven existence. Ma created a bank without any capital investment, and it became so successful that it was poised to debut as the most valuable listing on any stock exchange—until government intervention halted its progress. Ant Group's online bank, MYbank, extended credit to small and micro-businesses, traditionally overlooked by banks. It may well be the most successful example of 'fintech.'

A global revolution in financial inclusivity is unfolding, enabled by intelligent, automated systems that collect Big Data from across the entire business ecosystem. This data includes real-time, authenticated, and transactional data from financial actions, social media posts and interactions, customer and product profiles, and more about the business. Big Data enables a comprehensive approach to provide personalised economic and financial solutions.

For instance, small business borrowers in China applied for loans through their smartphones with just a few clicks and could receive cash almost instantly upon approval (the majority of applicants were approved on varying terms). No need for meetings with bankers, voluminous financial records, or references. The entire process took no more than three minutes, yet the default rate was less than 1%, and the cost of loan processing was just half a dollar. The online application and risk management system could collate and process over 2,500 pieces of information on each borrower within those three minutes. This intense and invasive yet effective use of 'legal/formal' access to Big Data of the private businesses is worthy of attention.

China's social credit system is a key source of 'credible' information on loan seekers' bankability, allowing for rapid assessments of creditworthiness. Social credit is a national 'trustworthiness' rating system for individuals and corporations. Initiated in 2014, it envisaged a six-year plan to build a system to rewards actions that foster trust in society and penalizes those that undermine it. What makes the system unique is that social credit is built on two types of information: traditional financial creditworthiness and 'social creditworthiness,' based on a larger swathe of everyday living.

Big Data Can Power Fairness

In an article for Mint, Jun Lou of Bloomberg reported a case involving the use of social credit data. The article highlighted how a small business owner might struggle to secure a loan due to a drop in their social credit score, which could result from something as trivial as failing to return a borrowed umbrella. Digitally collecting such information in a public database is yet another example of Big Data's far-reaching impact.

The bottom line is a true positive-sum game: healthy profits for MYbank and higher top line and bottom line for the borrowing small businesses, with millions of entrepreneurial dreams becoming reality.

The Repositioning of Innovation

Innovation is now the strategy! It's no longer just an ideal goal but the core business itself. Products and services are merely targets of innovation while being innovative and inventive has become the most critical strategy for organizations to survive and thrive in the future.

Big Data is reshaping innovation in several key ways. It brings together unstructured and legacy data (such as paper records and images), 360° data views, and contextual auto-recommendations. It also drastically reduces the cost of prototyping, enables co-creation with users, and scales innovation to the masses, reducing collective waste and driving wealth democratization. Innovation must now also account for the knowledge created by Big Data, as well as by 'artificial persons' like bots, robots, and IoT devices.

Big Data is now an organisational soft infrastructure; the fuel for the fire that is innovation!

Everyone a (net) producer, dignified

This is my favourite instance of how humanity is mathematising. The technology-driven era, dominated by mathematics, is also the most 'human' of all time. Mathematised intelligence particularly Artificial Intelligence is powering a truly human revolution. And, no matter the challenges, this unprecedented revolution will materialise. It is grounded in the mathematised human mind and the reform of mathematical education to ensure that everyone succeeds in the current K-12 curriculum (trust for now that the "how" is simpler than the simplest.) It is predicated on augmenting human intelligence with artificial intelligence.

Thanks for reading Cent Percent Mathematics! Subscribe for free to receive new posts and support my work.

Augmented (human) IQ – The Future of AGI

The effective, expansive, and explicit possibilities of complementing and supplementing our IQ with artificial intelligence is transformational in a few ways. First, it is here and now – humankind, as a whole, can close the widest disparity in educational opportunities for one and all, in a matter of years. The current formal education system is broken, and the only way forward is through 'artificial human' solutions, such as enabling a personal, 'model-learner' academic peer for everyone. However, it must be noted that current technology applications in education (Edtech) are fundamentally flawed—they are part of the problem, not the solution. The specifics of this issue, however, are beyond the scope of this post.

Second, and more importantly, the rise of AGI (Artificial Generative/General Intelligence) sends a loud and clear message about what 'Real Intelligence' (humans) must prioritize, especially in education: ensuring that every child and adult succeed in mathematics! Mathematics is the only domain of knowledge that can break the jinx of the vicious circle of poor education and put an end to both formal and informal education system that produces adults with suboptimal potential and capabilities. This is even more crucial now, as the rising cost of education and diminishing fiscal support make quality education, including preschool, increasingly out of reach for young parents around the world.

How is mathematics going to make a difference? Mathematics is a priori knowledge; we are all born with the foundational abilities for learning it—rationality and logicality. It must be 'taught' in the way we learn our first language, the mother tongue; primarily at home and within the community. In truth, mathematics cannot be taught in a conventional sense; it is learned culturally through lived experience. Expectedly, it is not human to be left behind in the language that is mathematics.

However, humanity is struggling with what is often referred to as 'school mathematics.' Whatever the K-12 system defines as mathematics does not represent the (real) mathematics, the language of the 'real world' (the language of physics – the most universal science, for instance.) K-12 mathematics is overly abstracted, rigourised, 'generalised,' and often 'meaning-less,' making it unfit for educating children, who learn best from 'near or the real/concrete, to far or the abstract'. To be fair, abstract mathematics is invaluable for engineers, technologists, scientists, and researchers, but it is not effective for learners (at all levels, perhaps.)

Magic of Mathematised Minds

To cut to the chase, "mathematics as a language" is the elixir, and the trailblazing snowball that leads to a virtuous cycle of multidimensional progress. This is especially so because AGI is already brilliant at methodised mathematics; it is in the mathematisation of real-world contexts where we can do better significantly for a long time to come. Indeed, a mathematised mind is best equipped to make the most of AGI, both today and for a generation to come. AGI represents the best news for the current state of humanity. Its outcomes—text, voice, video, code, design, plans, and whatnot—may serve as the foundational platform for human thinking and efforts, Setting a much higher bar for human productivity and achievements.

Moreover, all technological revolutions face significant challenges due to the necessity of re-educating everyone for the shift. Fortunately, AGI also serves as an incredible learning engine! It acts as both the the fuel and the fire. Each of us can "learn and live," endlessly, on the AGI soft infrastructure. AGI could very well be the (Terminal) General Purpose Technology (GPT) for humankind. A mathematised humanity is the game-changer, paving the way for a new sustainable future—socially, ecologically, economically, and politically!

4IR – The Economic Face of AGI

AGI is inching towards becoming the fulcrum of the Fourth Industrial Revolution (4IR) —a fast-consolidating new economic system. As a force multiplier, AGI represents an unmitigated blessing for individuals, societies, nations, and humanity as a whole. If AGI falls short of its promise, it is due to the lack of "worthy, competing humanity." Our education system is producing adults who are significantly "under par" compared to AGI's current capabilities. Furthermore, AGI will face strong resistance if we do not align our education system with the reality of AGI in our midst.

The roots of AGI in 4IR are democratising access to essential socio-economic soft infrastructure, including education, health, and ease of doing business. Humanity has never before extended such a welcoming hand to all. This marks the most significant difference between the Third Industrial Revolution (3IR) and the Fourth Industrial Revolution (4IR)—a shift from a top-down to a bottom-up approach.

Most notably, 4IR "softens and virtualises" the fabric of value creation and production fabric. Economic resources and opportunities expand seamlessly, capitalising on and cultivating individual productive instincts and cultural propensities. Strikingly, the DNA of 4IR is as biological as it can be—it represents a double-helix economic miracle that combines two distinct models: the proto-industrial system (which involves large-scale production without the traditional "factory model," powered by independent "Own Account Enterprises") and the intelligent-industrial system (a post-factory model with globally networked "Own Account Enterprises").

Every Family an Economic Sovereign

Undoubtedly, the universal unfolding of the Fourth Industrial Revolution (4IR) may still besome time away, but it is most eagerly awaited for the effective reinvigoration of "Own Account Enterprises," where every adult serves as a producer, contributing economic value. No one is left behind and designated solely as a consumer; rather, every adult becomes an economic sovereign. Ensuring economic surplus for every adult is the wellspring of dignified living. Even the poorest individuals assume multiple social roles—such as spouse, parent, child, sibling, neighbor, and citizen—and each of these social constructs, identities requires some level of financial means to fulfill, helping to keep the society duly oiled. Recall, society is a biological necessity for every one of us, not just a desirable concept.

Economic Dignity Guaranteed

Indeed, guaranteeing human dignity stands as the most revolutionary promise within the realm of possibilities offered by the 4IR. The vision of a world with over 8 billion+ dignified individuals surpasses our current imagination. It encompasses the broad socio-economic integration of all humanity, bridging divides between rich and poor, rural and urban, men and women, developed and developing nations, and other cultural and economic divides. Given that the 4IR represents a highly quantifiable transformation, achieving dignity for all can be mathematically structured.

Human Dignity Rests on Economic Dignity

It would be no exaggeration to see the idea of dignity as inherent to being Homo sapiens. Yet its universal recognition has been relatively recent, notably enshrined in the 1948 United Nations Universal Declaration of Human Rights. However, in contemporary times, the concept of human dignity is under severe scrutiny, being challenged and violated like never before, even as we consider ourselves the "best educated" humanity in history. Reports of abuse, violence, discrimination, humanitarian crises, and authoritarianism are prevalent across nations.

Nevertheless, human dignity encompasses far more than the mere absence of these adverse conditions. We are gradually coming to understand how deeply it is rooted in the economic empowerment of individuals. We must rectify this situation or risk living in the indignity of a subhuman existence. Worse, our acceptance of indignity for the majority could become an irreversible and unredeemable condition.

It is important to remind ourselves that governments—unfortunately, the only functioning human collective—cannot solely sustain the dignity of the middle class, let alone provide adequate support for those living in poverty and deprivation. Developed nations are already struggling with unbridgeable fiscal deficits, which has resulted in their per capita social investments dwindling. Many developing nations have already reached the brink of fiscal collapse. For instance, the Australian state of Victoria recognized for its social liberalism, announced the withdrawal of its bid to host the 2026 Commonwealth Games, citing insufficient financial resources.

A Common Market for 8 Billion+

Humanity must conspire to leverage technology as never before. We should collectively strive for the emergence of "Human Businesses"—next-generation enterprises dedicated exclusively and profitably to serving the common market of 8 billion+ people (similar to the European Union's common market). These businesses would focus solely on creating products equally valuable to all 8 billion individuals, eliminating the need for 'bottom-of-the-pyramid' products; intensely reconnecting rich and poor through economic integration.

The crux of it all - individual dignity cannot exist without assured economic dignity. AGI and the Fourth Industrial Revolution (4IR)

may be our best opportunity to realise 4 the beyond-dream 'peaceful, prosperous, and (continuously) progressive' humanity.
Universal human dignity requires mathematising humankind.

A twist to the tale – We all use deduction, all the time
Signs indicate that mathematical and logical thinking is inherent to humans as a priori knowledge – no specific learning or training is necessary to deploy it in everyday situations. Academic application, however, might require formal education. This innate ability naturally grows as we go about our daily routines. We are already somewhat mathematised, awaiting further honing and expansion through formalisation. The challenge lies in our dependence on the formal education system – curricula, textbooks, assessments, and teachers – which does not systematically introduce the thought process of deduction (and how it contrasts with induction, the scientific thought process).

At a rudimentary level, evidence suggests that we share a sense of quantity with certain animals. Most animals exhibit an understanding of their physical capabilities, for example, they displaying a sense of assessment of the length they can jump over with ease and do not attempt jumping over a wider drain. Similarly, research indicates that some animals, like crows, can differentiate among 1, 2, 3, and 4 quantities of something.

Deductive reasoning is an authentic, powerful mode of thinking about conditions and situations, consistently applied in our daily lives. For instance, based on the boss's past behavior of consistent lateness to meetings, I deduce that today's meeting would not be an exception. Hence, I might arrive late without consequence.

Observing that most questions in recent exams were from six out of ten chapters in the syllabus, I decide to focus solely on those chapters for my preparation.

Noticing a decline in orders for a particular product over four months, I conclude that the company needs to invest in new products. Recognizing data sciences as the fastest-growing career, I plan to transition to become a data scientist.

However, deductive reasoning is not always correct due to flawed premises and overgeneralisation. For example, Ill-informed premises can lead to incorrect deductions. For instance, assuming that all green-leaved plants need sunlight, and therefore a red-leaved plant does not, (and it could be kept wholly indoors). This overlooks the fact that need for sunlight is not solely due to colour of leaf, but due to the presence of the green coloured pigment chlorophyll in such leaves. The red leaves also have green pigment chlorophyll but that is masked by overwhelming presence of red pigments in those leaves.

Misguided assumptions also affect decisions. For example, we are familiar with the fact that objects in an open space gradually cool down to reach the ambient temperature due to the dissipation of heat from the warmer objects. It is commonly believed that to maintain the warmth of a liquid for an extended period in a room, we need to heat it to its boiling point and then cover it. This is a typical practice to sustain warmth in an open setting. However, this approach is flawed. The speed at which heat is lost to the surroundings depends on the temperature difference between the liquid and the surrounding environment. The greater this difference, the faster the liquid cools to align with the ambient temperature. Moreover, the rate of cooling is significantly accelerated when this temperature gap is higher. It is essential to understand that all objects radiate heat in proportion to the fourth power of their temperature.

Overgeneralisation can oversimplify complex situations and can lead to inaccurate conclusions. Overgeneralised idea that rural folks are inherently 'simpler' and 'sorted', hence, Aman, a recent migrant to a city, must also possess the same attributes of being 'sorted'. Assuming that because ABC is deemed the cleanest and greenest city in the country according to a recent survey, my friends residing in a colony in this city must automatically be experiencing a high quality of life.

Deductive reasoning is akin to solving a puzzle. Like puzzles, it requires a trained and informed mindset to solve. The process involves collecting all information about how to solve puzzles and applying it to the specific puzzle at hand. This similarity between deduction and puzzles makes deductive reasoning a personal skill. Notably, detective work heavily relies on deductive reasoning.

Generally, deduction involves recognizing and applying a set of broader truths, assumptions, or principles to specific situations in order to arrive at the most favorable decisions or actions guided by this comprehensive framework. The effectiveness and success rate of detectives heavily depend on the thoroughness and comprehensiveness with which they gather all types of information and evidence, without prejudice to the perceived value of the information. Subsequently, they apply deductive reasoning to the facts and evidence to narrow down to the specifics of the case.

It is interesting that many of us exhibit a strong and predominantly accurate intuitive and commonsensical approach when responding to emergent situations. The utilization of a subconscious logical, deductive reasoning process is undeniable in such scenarios. This is why deductive reasoning is a form of thinking prowess that cannot be easily artificially created and routinised.

To better comprehend deduction, it is essential to juxtapose it with induction. In short, induction, often referred to as the scientific method, is the process by which research proceeds to discover new scientific knowledge. Hypotheses validated by adequately repeated 'specific experiments' are utilized as general principles (laws) of science. In a sense, induction follows a bottom-up approach, while deduction follows a top-down approach to accumulating knowledge.

However, it is through deductive reasoning, often termed the 'mathematical route,' that many scientific mysteries are uncovered.

For instance, the expansive quantum field theory, purportedly regarded as one of the most comprehensive physical theories of all time, might await the development of the precise mathematics needed to unlock its secrets. Robbert Dijkgraaf, a mathematical physicist and the minister of education, culture, and science of the Netherlands (appointed in 2022), strongly advocates for the omnipotence of mathematics in understanding nature. He asserts that the workings of the universe follow an ordered and uniform mathematical structure, taking a bolder stance by suggesting that a proper mathematical comprehension of quantum field theory could potentially provide solutions to numerous unresolved physics problems.

In summary, mathematical reasoning—deduction—is a common practice among all of us. It is a matter of formally acknowledging, encouraging, and refining what's already a part of our lives. Humankind is already at a certain level of mathematisation, and mathematised mathematics autonomously raises that bar.

Another twist to the tale – Mathematisation of 'social sciences'

The conversation up to this point must not imply that the ongoing trend of increasing mathematization is confined solely to science and technology. Despite living in highly science and technology-driven times, our future is equally enriched by the mathematization of socio-cultural aspects of life and work. To this extent, STEM-focused research, innovation, and businesses position themselves as self-appointed guardians of humanity's well-being, yet many recognize it as a mere facade.

The increased mathematisation of social sciences research is not a new phenomenon. The utilization of mathematics by what we term 'social media' is widely known. It is notably sophisticated and continuously refined and restructured. For instance, platforms like Facebook employ Big Data and algorithms to dynamically tailor the display of pages and content to individual users. Each action of the user—clicks, likes, shares, friending, comments, and tags—contribute to and refine the Big Data related to the social behaviour of individuals, diverse communities, businesses (via their pages), and various other dimensions. These platforms utilize intricate, constantly evolving algorithmic tools, such as Affinity score, EdgeRank, Time decay, and Edge Weight.

Socio-cultural structures are incredibly diverse across societies, making it impossible to categorize and encompass any specific set of guiding principles for the mathematisation of social innovations. Comparisons can hardly be made between the pace and complexities of economic innovations. However, within the sphere of socio-cultural life, there exists a unique potential for a tectonic shift in the quality of our lives—mathematising humankind for true democracy.

Political philosophy, institutions, processes, and practices hold significant implications for societies. The mathematisation of governance institutions, predominantly in certain segments of the executive, often referred to as e-governance, was anticipated to drive us towards a more robust democracy. However, evidence from various parts of the world indicates a burgeoning executive that stifles the voices of citizens and opposition, conducts intrusive surveillance on numerous fronts, and fails to uphold airtight privacy provisions for citizens, community organizations, and businesses.

Mathematisation is sine qua non for true democracy, yet it would never be sufficient for a nation if e-governance were the singular focus of the mathematisation of the political society. The most groundbreaking consequence of mathematising humankind lies in the democratic revolution within each country. Furthermore, mathematisation should guide us in establishing resilient, true democracies, a first-time opportunity for humanity. At the core of the unprecedented 'poly crisis' facing us lies the political and moral crisis—the failure of democracy.

Nurturing and sustaining true democracy involves numerous dimensions. Considerations include what constitutes true democracy (as opposed to our current state), why we might seek it (its potential advantages), and the feasibility of its realization in current times (whether such a political revolution is attainable). In this context, the mathematisation of democracy stands as our sole hope to position society and citizens at the helm and in control of governance institutions, policies, and laws.

Yes, e-democracy presents a distinct form of political organization within societies when compared to e-governance or e-governments. The latter, in its current form, restricts democracy by implementing unparalleled, comprehensive, and intrusive surveillance of citizens and societies in real-time. There exists an unhealthy and menacing imbalance in the information and information flow between citizens and government institutions and political parties. To establish the technological foundation of true democracy, the essential component is the mathematisation or logicalisation of core democratic processes. E-government will play a role, albeit in a manner that serves democratic objectives. *Mathematisation of socio-cultural aspects of life is set to deepen.*

The Rising Linear Algebra – A Conspiracy?

There is an interesting twist to the story of mathematics, us, and destiny! An inexplicable development seems to have turned everything upside down, as if the universe—and perhaps something beyond—has conspired to force mathematisation upon us all, once and forever. To me, this is humbling and presents conclusive evidence of how humankind is singularly mathematising.

The must-know mathematics in the era of AGI (Artificial General Intelligence) is now accessible to everyone; it has become the 'mathematics of the masses.' A good grasp of foundational K-12 mathematics has emerged as a fundamental gateway to unlocking the best possibilities AGI offers—whether for personal, social, community, or professional purposes.

Linear Algebra – the easiest real-mathematics

Mathematics has always played a central role in the story of life on Earth, and now, luck seems to favour it as its most approachable and practical side takes centre stage in its applications. Linear algebra, often considered one of the more accessible branches of mathematics, forms the fundamental framework for mathematical modelling in the development of AI applications. As its name suggests, it operates within the realm of linear mathematics. Therefore, all equations studied in this field are linear, meaning the variables are used in their 'native forms'—raised to the power of 1. For instance, the equation

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$ represents a linear equation

Where a_1, a_2, ..., an, and b are constants, and x_1, x_2, ..., x_n are variables raised to the power of 1.

Being algebraic, linear algebra quantitatively expresses relationships. A given equation represents a unique relationship among the variables (varying quantities) it relates. For instance, the equation $5x + 2y = 9$ expresses a particular linear relationship between the variables x and y. Here are a couple of examples of the possible interpretations of this equation:

- The total count of items, if there are 5 single items and 2 pairs of the same items, is 9.
- The total count of people, if there are 5 single people and 2 couples, is 9

Countless other situations could fit this relationship. In general, a linear equation signifies that x and y can only change in a specific manner; every concurrent pair of values for x and y must always adhere to the given relationship.

We 'solve' such relationships to find the specific values of x and y that satisfy the relationship in the equation. The 'solution' for a linear equation means there is only one value for each variable. For instance, in the equation $5x + 2y = 9$, there is just one specific value of x and y (x = 1, y = 2) for which 5x + 2y = 9. This solution corresponds to both of the examples above.

In fact, any situation involving more than one simple variable requires the use of linear algebra to mathematically articulate and utilise it as a generalised model of relationship.

Big Data and Linear Algebra – Made for Each Other

Proficiency in understanding and employing linear algebra tools form the foundation for effectively utilising 'big data' to creatively tackle a wide array of scientific, technological, social, economic, and even political or governance challenges and objectives.

Linear algebra empowers us to envision, interact with, and manipulate n-dimensional scenarios—whether scientific, technological, social, and more—where each dimension represents a different variable. These variables collectively define the scenario. This illustrates why abstract mathematics is so powerful and appealing: its applications are limitless and expansively versatile.

Most people find it difficult to comprehend anything beyond three-dimensional space or objects. However, some—especially physicists—have ventured into visualizing four-dimensional space-time combinations. While visualising higher dimensions may seem abstract, it's not far-fetched to imagine scenarios in n-dimensional space, depicted as ordered data involving a list of n variables. This is one of the reasons linear algebra plays such a vital role in handling complex, multidimensional data in big data analytics.

A simple application of linear algebra would be estimating the price of a house in a city based on several features or variables, such as neighbourhood, number of rooms, floor area, floor plan, municipal law, landscaping, construction quality, amenities, etc. We would collect data on these variables for a lot of houses, along with their prices. This creates a multi-variable or multidimensional dataset, allowing us to predict the price of a house based on its specifications across these various factors. While we often use tools like Excel to perform these calculations, we tend to overlook the underlying linear algebra that drives these price outcomes.

In our pursuit to comprehensively understand and model increasing volumes of data, we frequently augment the number of variables during data collection. Greater intelligence necessitates the inclusion of more variables. Consequently, making linear algebra grow in potency and usefulness for constructing progressively intelligent devices and systems.

Linear algebra serves as a fundamental analytical tool for various systems that are increasingly being embedded with intelligence—such as engineering (e.g., analysing the dynamics of flow in a network of pipes), economics (e.g., understanding price, supply, and demand dynamics), science (e.g., weather forecasting), and consumer products (e.g., managing the toothpaste portfolio of a large consumer company).

Linear algebra allows us to model, comprehend, and manipulate systems of equations involving numerous dimensions or variables. For instance, face recognition software heavily relies on linear algebra. It organises facial feature data into massive 'pixel-by-pixel' matrices, where each pixel represents a specific feature of the face relative to a standard. Linear algebra helps with data processing (compressing the data without loss of integrity), analysis (through matrix operations), and manipulation (training models and extracting features, again using matrices).

The reassuring bit

It is encouraging to realise that linear algebra is part of school-level mathematics. There is no reason why anyone should struggle with mastering it, except due to the quality of mathematics education at the school level and a lack of rigour in the faith and belief of the educators in ensuring all students succeed.

In contrast, the application of mathematics in physics demonstrates a relatively stronger command over mathematical principles. While calculus computations might not be suitable for everyone, Linear Algebra, in comparison, is accessible to all—it is much mechanical, and simpler; though computational calculus is also overwhelmingly algebraic and trigonometric. Even theoretical physicists tend to favour more familiar and 'simpler' mathematics. For instance, consider the Heisenberg uncertainty principle and the Schrödinger wave equation, both independent theories in atomic physics. They share a similarity in asserting that a more precise determination of an atomic particle's position would compromise the certainty of its momentum. Essentially, their mathematical formulations are alike. However, Schrödinger's equation gained more popularity as it relied on more familiar differential equations.

Overall, the rise of simpler mathematics is helping to accelerate the mathematisation of humankind. The central role of linear algebra in Big Data applications is a clear sign of the massive expansion in the appreciation of mathematical ideas and objects, leading to a more precise understanding of science, technology, engineering, and research.

The increasing use of 'simpler mathematics' is like a backdoor entry of mathematics into our lives—it's subtly permeating everything we do!

The increasing applications of 'simpler mathematics' is like a backdoor entry of mathematics in our lives—it is subtly permeating everything we do!

Linear algebra and its applications

Linear equations are notably effective in approximating real-world situations. One intriguing scenario involves quantities that require multiple dimensions or variables for complete definition or understanding. The simplest among these are quantities known as vectors—they possess one dimension as magnitude, akin to scalar quantities, while the other dimension denotes their direction of change. For instance, speed is a quantity defined by a single dimension known as magnitude. However, when considering speed along with another dimension—direction—it becomes velocity. Thus, velocity represents a vector quantity with two dimensions: magnitude and direction.

In linear algebra, vectors hold a fundamental position, playing a central role in various key concepts and techniques. Using vectors in linear algebra offers a significant advantage—they provide a robust and adaptable method to represent and manipulate complex quantities. Vectors can undergo addition, subtraction, scaling, and various transformations, which can be combined to create more sophisticated operations and structures. Consider an airplane landing, a scenario influenced by several vectors: wind, drag, wing and tail positions, along with Air Traffic Control (ATC) instructions guiding pilots on a specific heading (direction) for a set distance (magnitude).

A matrix, much like a vector, comprises a collection of numbers, while linear transformations encompass the set of all functions (functions that take vectors as inputs).

In linear algebra, matrices serve to represent linear transformations and are expressed through matrix multiplication. For instance, the rotation of a 2D image on a computer screen exemplifies a linear transformation, which can be represented by matrix multiplication.

Linear algebra holds significant connections to various areas of mathematics, notably including probability, calculus, and statistics, because it provides an efficient means to represent and manipulate data. Its role in statistics and probability theory is particularly crucial.

In statistics, data is frequently organized in matrices or vectors, where each row signifies an observation or data point, and each column denotes a variable or feature. Operations in linear algebra, such as matrix multiplication, are instrumental in conducting computations on these data structures.

In probability, regression analysis is a statistical technique used to model the relationships between variables. Linear regression assumes a linear relationship between the dependent variable and one or more independent variables. The coefficients in a linear regression model can be estimated using techniques such as ordinary least squares (OLS), which involves solving a system of linear equations—a core concept within linear algebra.

In calculus, linear algebra is used to study the functions of multiple variables and their derivatives; linear algebra facilitates the solution of linear systems of differential equations.

Linear algebra can also be used to study optimization problems, which involve finding the maximum or minimum value of a function subject to certain constraints.

It may be encouraging to realize that linear algebra is part of school-level mathematics. There is no reason for anyone to struggle with mastering linear algebra, except due to the quality of school-level mathematics education and a lack of rigour in the faith and belief of the educators in ensuring all students succeed.

In contrast, the application of mathematics in physics demonstrates a relatively stronger command over mathematical principles. While calculus computations might not be suitable for everyone, Linear Algebra, in comparison, is accessible to all—it's simpler. Even theoretical physicists tend to favor more familiar and 'simpler' mathematics. For instance, consider the Heisenberg uncertainty principle and the Schrödinger wave equation, both independent theories in atomic physics. They share a similarity in asserting that a more precise determination of an atomic particle's position would compromise the certainty of its momentum. Essentially, their mathematical formulations are alike. However, Schrödinger's equation gained more popularity as it relied on more familiar differential equations.

On the whole, the central position of simpler mathematics would spur the faster and wider mathematisation of humankind.

Mathematising mathematics – The pearly gates of education

This extensive topic is reserved for a later chapter in the book. The 'technology of education' stands as humanity's blind spot, revealing our struggle to comprehend the means to nurture an infant toward reaching even 'half of their human potential.' Unfortunately, it seems to reflect a race to the bottom, as exemplified when a U.S. president famously urged teachers to compete against Indian children in mathematics, despite the puzzling dilution observed in mathematics education in India.

We have misconceived education to the extent that Edtech is now hailed as 'Technology in education;' expanding technology's presence in a domain that is fundamentally social. Education relies on role model adults, peer interactions, conversations, observations, experiences, and the development of habits of both body and mind, such as reading and writing.

Furthermore, the intensifying institutionalisation of 'educating children' might be humanity's most significant misstep in the past 200 years. The advent of the Fourth Industrial Revolution (4IR) will inadvertently lead to the de-formalisation of education, returning it to the domain of parental guidance, family influence, and the broader societal community 'the village'.

The growing public apprehension towards AI, urging for regulation to 'combat it,' actually signifies our collective failure to grasp the core of education. It is high time we equip people to align with AI, prompting a redesign and revolution in education. To cut through the complex context, the educational revolution hinges on the lack of a just any one domain of knowledge, skill, value, or attitude that every school can effectively instill in all its children, without exception. Currently, we are attempting numerous initiatives, all falling significantly short of the mark!

There exists only one such domain – mathematics! Unfortunately, K-12 education shows the poorest possible record of achievement in mathematics. K-12 has yet to de-arithmetise mathematics, and view it as the language of the gods and the universe, the language intertwined with everyday life. The philosophical foundation supporting mathematics as a natural-like language dates back several centuries. David Sepkoski, from the University of Illinois at Urbana-Champaign, in his research on seventeenth-century mathematical philosophy, suggests that 'the epistemology of mathematisation is fundamentally linked to the epistemology of language.' Epistemology refers to the *'philosophical theory of human knowledge.'* For instance, the previously mentioned 'epistemology of language' could be interpreted as how we acquire and master a language.

Mathematising mathematics is non-abstracting mathematics. K-12 reassertion will start with mathematised mathematics.

Without further ado, let us just say that humankind is mathematising as K-12 remakes itself.

Regulating AI – A lame debate without 'making men'

Regulating the AI industry is an ongoing, contentious battle. Surprisingly, some industry leaders advocate for seeking regulation while simultaneously advancing their vision for AI platforms and products. It is premature to firmly adopt a position on regulation or delve deeper into its evaluation at this stage.

The crucial and fundamental issue concerning the progression or containment of AI is the current and future nature and level of organic intelligence. The interaction and relationship between humans and AI depend on the master's capability and how we strive to surpass and maintain superiority over AI—continuously growing to maintain our mastery. Human capabilities are boundless, and our only limits lie in how we educate ourselves, determining our individual and collective virtues and potential.

There is little debate when it comes to advancing the mathematisation of humankind, regardless of how or when we regulate the AI industry. Mathematisation also involves rejuvenating society by empowering its basic units—individuals and families. Society represents the unseen force and structure in our lives. It revolves around instilling the concept and guarantee of social welfare, allowing genuine democratic control over our shared destiny.

We must realise that debate on AI regulation lacks foundation, and is without considering how humanity will progress in the future. In fact, AI itself plays a crucial role in facilitating the ability of humankind to retain best control over AI.

Furthermore, without striving to ensure the mathematisation of the entire human race, the debate about regulation falls into the hands of a fraction of us who may have a better understanding of AI but cannot genuinely represent the best interests of all of us, or the potential collective advancements in the realm of AI.

It all comes down to 'what it means to be human', 'what defines our humanity', and how mathematics serves as the fundamental stepping stone to understanding 'what makes us human'. As a corollary, it raises questions about 'the essence of education', 'the connection between the education system and our humanity', and 'the role of mathematics in education'.

Section II

Being Human

Being human has never been more debated than the current times. Maybe, this is a reflection of the marginalisation of society in our lives, we seem to be losing that society is a biological need, not a 'desirable, ideal need, or creation.' It has led to the loss of cultural life-force in our identity. The dramatic loss of faith and presence of the '(western) liberal democracy' also brings us to question if long-cherished human values are really to die for.

Fortunately, help is at hand to investigate the state of human ideal. Horace Mann, a leading founder of the K-12 system, famously declared, *"A human being is not attaining his full heights until he is educated."* Indeed, it all boils down to how we educate newborns to adulthood.

Emmylou Harris, regarded as musician's musician and a political activist, says it all about education in asserting that *"Animals have a much better attitude to life and death than we do. …"* No animal, except for humans, needs to be educated to be what they are expected to be (as an adult). We are only as human as our education.

Celebrating life is a right

Born to live king-size

All living beings are born truly themselves, there is little to be discovered, learned, or thought about. A pup is born to be a dog, it simply increases in size as it matures. A calf will grow to be an elephant, no matter what. Their environment has little to do with their being, their innate nature. They may be 'beaten' into submission/conditioning or lured into one (such as Pavlov's dogs) but it does not change their reality one bit.

We cannot even change the character of single-celled (bacteria, and more), or part-celled acellular (virus) entities; except for changing the very nuclear material, but in that case what we have is a new living being (new genetic makeup is new kind of living being, by definition). However, this truth may be the biggest celebration of what living is, how life is in complete contrast to being 'dead', and why we have not been able to create life as yet! That plants grow as the seeds dictate is also well known, the secret of every plant is hardcoded in the seeds.

The best sameness is apparently boring! Diversity is the true king.

Zero is hero

No wonder there is all the similarity between animals (and plants) of the same kind/species; we all can narrate several universal features of all dogs, all whales, all sparrows, all lions, etc.

Are we, humans, an exception to all this? Yes, no two of us are (apparently) alike, in the present, as well as longitudinally – the remotest past and the forever future. Why are we so exceptionally unique? The answer to this quest is even more extraordinary – we are all born 'zero', a clean slate, there is very little human about us at birth (except physiology, of course)! This tabula rasa is the secret of unbounded human plasticity and (thriving) population.

Diversity cannot feed on unlearning, but learning.

Twenty-first century heroes that children are

The yet inexplicable 'spark that is life' is most sharply visible in the survival instinct. It is so hard to die, every life is a spring of 'life force'; dead plants sprouting out shoots are quite a common sight. Obviously, human infants must have such force in most abundance, and it is.

Every human infant is born ready to learn! Ready to interact with her environment, all set to make more sense of the world. The ability to learn, and think (not reflex) is a natural corollary of the 'nullhuman' that every infant is. It is as easy to raise a child to be a Christian as a Hindu, a Dane as a Malay, a musician as a mathematician, and all else.

To top it all, it is as if the endless universe has conspired to ensure a unique phase of growth for humans – childhood, which ensures the best learning capability to every infant by being devoted to the development of the brain with the body getting secondary attention.

Infants know how to learn, we do not need to teach them that (even if learning can be taught); for instance, can we really teach children how to start walking!

Last but not least, the near 'universal institutionalisation of raising children' (K-12 to University now 'educate' our children as the primary and sole social institution) first kills the innate ability to learn, and in the best institutions we try to 'teach how to learn' – a complete contradiction in terms. We will further explore these themes later in the paper.

The knowledge era, the AI-age heralds the best time to be human.

Humans have to be raised to be so

Of course, we know this for eternity – it takes a whole village, to raise every child in the village. Raising a human is intensely, and organically social, and 'local'. For, our existence is comprehensively (and biologically) rooted in our being a part of a society. Our most important identity, especially in the formative years, comes from the values, morals, and resources we are immersed in, it is our Social DNA – our real existential DNA.

And then there is family for every one of us. The 'clean slate' newborn, physical limitations of children, infinite things to learn, and the critical showcasing of the space to be self within a society imply years of 'nesting' for every newborn. Raising children in a social unit family is among the most creative human innovations, and also the most amazing gift for humans (to be role models to their children if no one else).

Down the lanes of history, societies also created special institutions – schools, colleges, universities, and many more of the kind – for raising children to fuller human potential and to even out the family differences in some socially important matters.

The nesting period for human children is about two decades; the time it takes to lay the necessary foundations to raise a human out of a newborn.

It should not be disheartening to realise that no one can be raised to be a full-human (the similar, 'vertical' development), it is as yet undefined, and we all know umpteen examples of the infiniteness of human ability along specific human dimensions. Yet, implicit in the aforementioned is the unfathomable 'lateral development' possibilities, i.e., in the way we all could be unique, and serve self and humanity in enriching ways. There is so much to be human, and for that matter being 8 billion should be a blessing for humanity IF we all are unique if we have been truly educated over the two decades to adulthood.

Education is the ONLY business of societies. Education is the technology of humanization.

Education is the name, game

This peerlessly human, two-decade-long process and content of raising children is what we call education.

Infant + Education = A cultured adult; to be right, culture also subsumes sci-tech

Jumping the gun, one of society's gravest self-inflicted hurt and one that explains all of the banes as well as boons is the near institutionalisation of education. Empowering academic (social) institutions — K-12 to University, examination and curriculum boards — and creating national ministries of education to define, standardise, assess and evaluate the education of every child is just wrong, untenable, anti-human, and anti-society. Unfortunately, this trend is only accelerating, and may already be viciously spiralling, and creating sub-humans (or unhumans, or humans with very narrow capabilities) at a mass scale even in the so-called best institutions. 8 billion of us may overwhelmingly be far fewer humans.

Education is entirely social, including the development of scientific temper. It must be rooted in relevant 'nature', the same nature we draw scientific principles from and therefrom technologies.

Family, community and society must be the natural (primary) educational space, and academic institutions must be shown their place as complementary and completely secondary. To be true, for once we need to be drastic about not associating with the phrases such as 'school education', college education', etc. Only purely social institutions should be acknowledged and encouraged as educational institutions.

In the increasingly sci-tech times, academic education is important to best understand the imperatives and opportunities open to societies. But it must be mediated, to some significant extent, through the adults in children's lives in family and society at large. A humanity where the best of the formally qualified parents cannot support even Grade III academic transactions is a wronged humanity and misplaced academic institutions. All academic education must be real, rooted in nature and population.

De-institutionalise education, educate parents, catalyse evolutionary revolution.

Rationality and Morality – the essence of education

Rational thinking and moral behaviour are the two universal characteristics of humans. Rationality is our biological DNA while moral behaviour is the expression of our social DNA. Rationality and morality together best describe our effective identity and actions.

Most simply, education is about honing rationality, and imbuing morality (ideally, rationally). On another plane, 'rationally presented morality' is a great tool at the hands of a society to steadily reinvent itself, and minimise the natural inter-generational dysfunctions.

The best educated are highly rational and moral.

Mathematics – Music to the ears of humanists

To be fair, between rationality and morality, rationality is more universal human character in the sense of its educational imperatives – it may be presented and processed in quite similar ways humanity-wide.

Retaining and sharpening rational thinking rigour and routines shall have an epoch-making positive impact on humanity as a whole. Developing, sustaining, and refining an educational system that leaves no one behind in harnessing the best of rational thinking is the biggest human innovation and the only solution for strongly emerging out of the current quagmire.

Obviously, it would be ideal if the rationality-focused education that guarantees equal outcomes for all children across the world supports the foundations of moral development too.

Mathematics education is comprehensively rational and independent of socio-economic, political, and technological divides. It has elements that could also rationalise the substrate for moral behaviour.

The role of mathematics in humanizing all children, and in due course, the entire humanity, is unprecedentedly revolutionary. It has never been visited for this potential ever before. In fact, the current math education has alienated and dubbed almost all humanity 'weak in mathematics'. Mathematics is the only aspect of our lives where we see no cringe in letting the world know how poor we are in mathematics.

However, all this is only due to 'school mathematics' not being mathematics at all! Human children cannot ever be struggling in 'cent percent mathematics, the real mathematics'.

Mathematics is the ONLY fuel to power 8 billion rational and moral us.

Cent Percent Mathematics – mathematics humanity lost out

'Cent Percent Mathematics', the real mathematics is a sense, much like the 'common sense', and the 'intuitive sense', an innate ability that cannot be taught, but is personally sharpened through experience, application, and articulation. Real mathematical sense grows with every problem or question solved (unlike the thoughtless 'practice' in what is called mathematics in schools). In fact, the other senses – the five – also grow through personal attention to them, for example, it is music that refines the hearing sense, the music we play or sing! Real mathematics is transacted like a language – we Read, Write, Speak, (and Listen) it.

The rational nature of mathematics is uncontested, though little internalised, and we start with a glimpse of the moral foundations and outcomes of mathematical constructs.

Real mathematics is the simplest language conceivable. More on this next.

Mathematics and humanisation

Google search 'Math helps in humanization?' and the response one gets is 'Humanising math classes', 'On humanization of mathematics', including 'Did you mean: math helps in ***immunization***'! This explains it all. Competent mathematical thinking and skills have never been seen as a universal human quality. Success in mathematics education is not a compulsory, non-negotiable educational goal. Thus, the only acknowledged challenge in mathematics education is 'humanising' it - humanising the disability in learning mathematics by systematically normalising poor achievement (such as lowering the standard of mathematics curricula, syllabi, assessments, evaluation, and policies), and reducing the shaming and supplementary education that almost always starts with mathematics.

No less, the role of mathematics in nurturing a fuller human is still an outlier interest in any kind of research. There seems to be no conviction in any domain of knowledge and practice, including philosophy, that mathematical thinking is a human imperative. Unfortunately, mathematics education has never been about humankind, about us all, even though since recorded history mathematical objects, thinking, and skills (geometrical, to be sure) have been the only thread connecting the entire humanity, besides the DNA material. That is the very nature of mathematics, being the language that captures the ways of our world (natural as well as man-made technologies), and the universe. Sunil Singh, a devout mathematics story teller and author, is apt in saying that, '*the humanness of mathematics has been almost surgically excised, leaving only a body of work to be picked over, rummaged through, and artificially assembled without coherence and historical context.'*

Today, mathematics education holds the key to how humanity will end up by the end of the century.

For, to date, we have no alternative to educating infants to be human (cultured, dignified) adults.

Mathematised Mathematics

Auto-mathematisation of humankind

De-arithmetisation, de-rigourisation, and de-formalisation is mathematisation, by default. Mathematisation is the promotion of intuition, visualisation and verbalisation in mathematics education. 'Reading mathematical texts', as opposed to 'doing' mathematical solutions, is an act of mathematisation. The abstraction of mathematics education dramatically increases the focus on arithmetical formulations. It must be known that abstraction is one key reason mathematics has such wide applications; '2 + 3 = 5' is an abstract mathematical expression, it may imply '2 cells + 3 cells = 5 cells', and '2 planets + 3 planets = 5 planets.'

However, placing mathematical expressions in real-world situations leads to real, non-abstract, mathematisation. For example, in everyday situations, we would encounter expressions such as '2 dozen cells + 3 cells', or '2 $100 bills + 3 $1 bills' and for neither of the sum is '5'; '2 dozen cells + 3 cells = 27 cells', and '2 $100 bills + 3 $1 bills = $203'. The highly abstracted mathematics is not wrong, but it represents just a fraction of the quantities that mathematical expressions may represent.

Very importantly, '12 + 13 = 25' is a memorised expression and it is only for those schooled and with memory of the sum, whereas '12 ₹10 bills + 13 ₹20 bills = ₹380' would be a universal response of all adults irrespective of their school education, or regular usage of mathematical computations.

Recall, in the entire education system, all the way to the masters level, 'word problems' are considered to be tough questions to understand, articulate mathematically, and solve to compute the desired quantities. In reality, 'word problems' (mathematics questions that are rooted in real-life situations) should be easier to solve correctly, than their corresponding abstracted problem. For instance, for a big majority of Grade V students, and many professionals, the abstracted expression '$\frac{1}{2}$ x + 5 = 100' may not be such a straightforward expression to solve for the value of x, but one of its infinite non-abstracted equivalent expression would be solved easily by all adults. For example, finding the cost of a dozen oranges (or $\frac{1}{2}$ dozen oranges) if 'Cost of $\frac{1}{2}$ dozen oranges + packaging cost (₹5) = ₹100'.

Understanding, manipulating, and applying quantitative relationships in everyday life is almost natural to humans. Fortunately, Big Data is real-world data, a record of actual, or simulated events and conditions. Reading and analysing Big Data necessarily demands the best knowledge of the context, data values, and the goals of the analysis. Big Data analytics auto-mathematises quantities and their relationships.

Let us explore an example of how the real context of data matters in any data analysis. Sheer application of arithmetical formulas may just be less than meaningful. The example is around the measures of the central tendency of data, especially mean and median values. It is a simple situation of how the choice and application of the two measures cannot be mechanical/abstract, the understanding of the nature of the data, and the purpose of computing the central measure are essential. The following example of population density (of a geographic region) is presented as a case study.

Should we compute the average population density or the median population density?

It all depends on the best understanding of the nature of data and the purpose of seeking the central tendency.

Commonly, average population density is a mean, that is, the sum of population densities of various regions/number of regions. However, to be mathematically sound, it should be known that density is in itself a ratio, not a mean value, and just the way a ratio is conceptually closer to division, density is computed using the division operation.

Pertinently, this computation of population density works well for small areas where there is no significant variation in density, but it often shows somewhat inexplicable scenarios for larger areas, where there are clear population variations, like downtown, suburbs, exurbs, and industrial areas. Thus, the average population density is hardly the experienced density for people who live in that region.

In such varied geographical areas, a more realistic measure of the population density would be the median value of the mean density of smaller or homogenous locations across the larger area.

The median value of the density is one which is more than the density experienced in 50% of the localities (not people) in the region, and less than the density experienced in the rest of the localities.

For example, consider the following data of the mean population densities of the five zones of a metropolitan city (in persons per square kilometre):

100, 200, 300, 400, 5000

The mean or the average population density of these zonal densities is 1200 persons per square kilometre. It is way higher than the density values of the first four zones because it is influenced by the extremely high value of 5000 (most likely in the downtown area).

On the other hand, the median population density is 300 persons per square kilometre, and it is apparently a better measure of the density across the zones and is the density that will be experienced in four out of the five zones.

Be real

Mathematics is NOT an arithmetical method, devoid of all logic!

Mathematics has been abstracted to make it widely applicable, and that does power math to become an eminent domain of knowledge. But it does not mean that we should teach math this way, i.e., abstractly where '7 – 3' is always 4! It is not. School math education makes math abstract a little too early, profoundly changing the very idea of math to children in a generation and adults in the next generation. Mathematics has been made out to be what it is not!

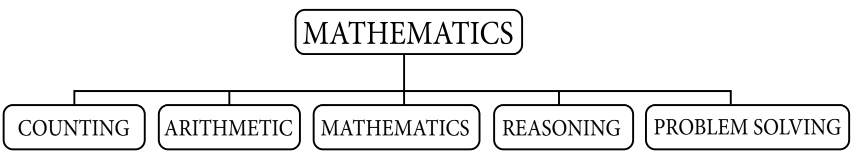
Five dimensions of math

The simplicity of the 'definition of math' in the Draft (Indian) *National Education Policy, 2019* is worth a reference. It defines mathematics as composed of the following five dimensions. To be particular, the elaborations are our own.

1. *Counting (and measurement)* The means and rules of quantification of things to generate numbers (the starting point of anything mathematical); for example, the whole idea and practice of counting number names, decimal number system, and fractions.
2. *Arithmetic* The basic rules for using and manipulating numbers. The idea and applications of the four fundamental operations on numbers; for example, how the quotient 4 in $\frac{12}{3}$ differs from quotient 4 in $\frac{16}{4}$.
3. *Mathematics* The language-like ability to think and express all quantifiable situations using numbers and arithmetic. For example, 'mathematics' distinguishes between '8 – 1 – 1', and '8 – 2' (there are significant differences between the two expressions)
4. *Reasoning* Math is uniquely logical and rigidly hierarchical. Anything mathematically expressed just needs to follow one other statement – the previous mathematical statement/

step. Unlike science, math is endlessly and purely logical. One can complete several PhDs in math with just papers and pens to write! For example, deductive reasoning is mathematically valid, while inductive reasoning is not. Science is observation, experience, experiments, and logic – all founded on what is real.

5. *Problem solving* The ability to harness the vast set of tools and resources in the domain of mathematics to define and solve any given real-world, scientific, or any research situation/event. For example, conceptualising a surface that has the maximum area for the given perimeter.



Hope you did register that one of the dimensions of mathematics is named *"mathematics"*. This just emphasises the core of mathematics – mathematical thinking. Not just number crunching!

Arithmetic of mathematics

This '5-dimensional view' of mathematics has some profound implications for teaching and learning math:

What we learn in the math period in schools is just one category of knowledge out of the five – arithmetic – and that too in a very limited manner: using specific methods without logic, and without the freedom to use different methods to solve the same problem (severely limited number of methods is also a challenge). It would not be too misplaced to say that math education in schools does not represent what math really is!

No wonder, the overwhelming majority of children in school struggle in math – their innate power of reasoning and language is crippled by the inappropriate school math content and emphasis on teaching rigid, limited arithmetic. For example, no reason is assigned to why '10 is written as 10', why '2 divided by $\frac{1}{2}$ is 4', or how exactly '125 percent of 80 is 100'. The right math education will logically explain every such mathematical expression/step.

Till the end of the 'whys'

It is heartbreaking to come to terms with the most undesirable impacts of math education – putting adults in poor light! After a few attempts at clarifying reason/logic, children realise the fruitlessness of seeking explanations from the adults around them – teachers, parents, elder siblings, and tutors. Everyone has studied math the same way – just rote methods-so no one can help a child get the answers to 'why' in math!

This tells us something unique about what math is. Math is the only subject that can take on endless 'why'. The real value of math is in honing a habit of thinking that seeks why for everything, all the time, and till the end of the whys. This habit is valuable across 'subjects of life'. It is my personal experience, not a poetic expression that math sculpts thinking reflexes and regimen, which significantly enriches team play and leadership, solution-seeking focus, and sharp communication abilities.

'School math' is logicless, thus unduly challenging

School math is anything but mathematics! It's not even arithmetic. School math is a creation of its own, invented only to be taught at school. Math learnt at school is of little use to those who find their ways to excel in math in school years (such as senselessly practising 'take 2 up to find the quotient of $\frac{8}{\frac{1}{2}}$ or $8 \div \frac{1}{2}$'), and also for those who get scarred by school math.

There is no reason for math to be taught in schools the way it is. Worse, school math lets the vast majority of children be branded as 'weak/slow learners' in school years, a fact no nation, society, community, or parent has dared to dispute and undo. Surprisingly, mathematicians have also failed to stand for our children and the fact that no child should be struggling in K–10 math (because math is based on logical thinking and needs no real-life resources/aids to be validated). School math is all comprehensively misplaced.

Here is another set of examples of the wrong that is 'school math'

1. 7 + 3 = (always) 10 according to school math! It is not so in everyday situations. For example,
   '7 years + 3 months = 87 months = 7.25 years = 2645 days' (assuming 365 days a year and all calendar months to be 30 days); there is nothing '10' about the sum of '7 years + 3 months'. School math discounts units of quantities – a cardinal mistake.
2. 2 × 8 = 8 × 2 and 3 × 4 = 3 × 2 × 2. For example, for many practical purposes, 2 packets of 8 apples are not the same as 8 packets of 2 apples. Math is a very precise language; physically, a × b is not the same as b × a. There is specific identification of multiplier and multiplicand (except when we use numbers as factors). A common manifestation of this indiscreetness is that we all do not know where to find '5 × 6' in multiplication tables – table of 5 or of 6!

Exemplifying 'school math' and mathematics

The stark difference between 'school math' and 'mathematics' may be visually examined in the simplification of the expression 6 × 2 ÷ 4 ÷ 3 × 5 – 5. We will simplify the expression in the 'school math way' and 'mathematically'. We all know the school math way – comprehensively abstracted simplification with numerals.

6 × 2 ÷ 4 ÷ 3 × 5 – 5

The school math way of simplification:

6 × 2 = 12

$$\frac{6 \times 2}{4} = \frac{12}{4} = 3$$

$$\frac{\frac{6 \times 2}{4}}{3} = \frac{3}{3} = 1$$

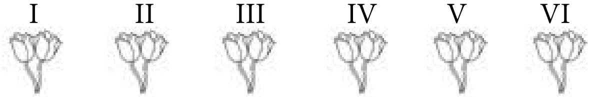
$$\frac{\frac{6 \times 2}{4}}{3} \times 5 = 1 \times 5 = 5$$

$$\frac{\frac{6 \times 2}{4}}{3} \times 5 - 5 = 5 - 5 = 0$$

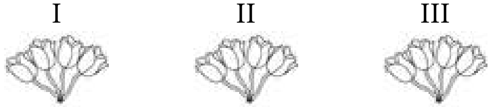
Mathematically, we will follow each step of the simplification in a non-abstracted way and visualise every step with quantities (numbers) rather than numerals.

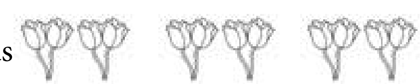
Let 1 =  then 2 = 

I II III IV V VI

6 × 2 = 

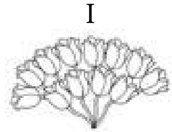
I II III

6 × 2 ÷ 4 = 

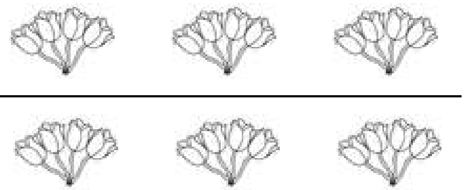
6 × 2 ÷ 4 need not be seen as 

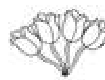
While 6 × 2 does represent 6 pairs, the pairing is not 'boxed/rigid' the way it is for the groups created out of division. Thus, 6 × 2 is still a collection of 12 things. Thus, 6 × 2 ÷ 4 is better represented as a collection of three groups of four roses.

I

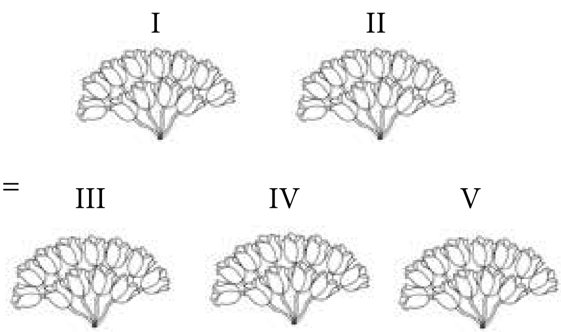
6 × 2 ÷ 4 ÷ 3 = 

$6 \times 2 \div 4 \div 3$ is packet(s) of 3 groups of 4 roses in each packet. And there is only one such packet of 3 groups of 4 roses possible out of the given 3 groups of 4 roses. Pictorially, $6 \times 2 \div 4 \div 3$ is:

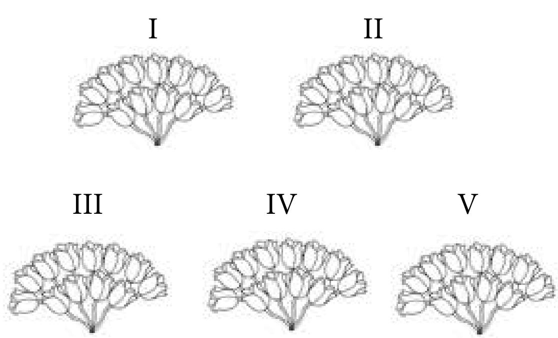


The divisor 3 in '$6 \times 2 \div 4 \div 3$' has to have the same unit as the dividend '$6 \times 2 \div 4$'. The unit of the dividend is , and there are three of those in the dividend.

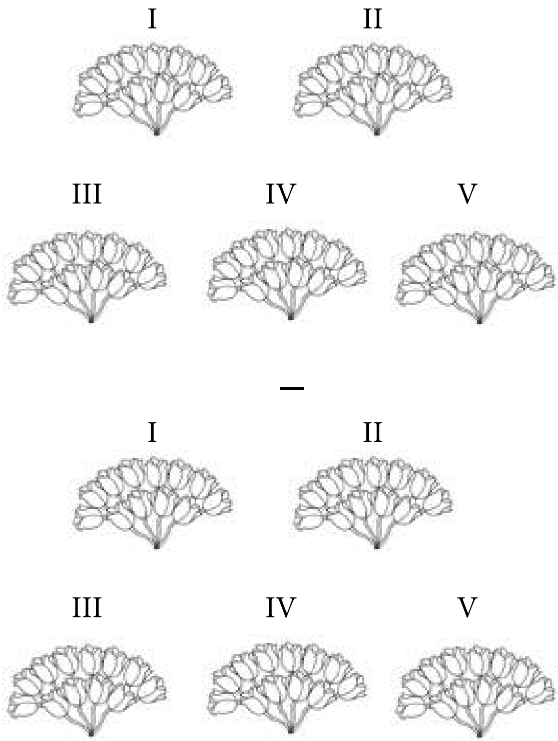
$6 \times 2 \div 4 \div 3 \times 5 =$



As the unit of subtrahend 5 in '$6 \times 2 \div 4 \div 3 \times 5 - 5$' is not specified, it is assumed to be the same as that of the minuend,
'$6 \times 2 \div 4 \div 3 \times 5$', hence minuend 5 is:



6 × 2 ÷ 4 ÷ 3 × 5 – 5



6 × 2 ÷ 4 ÷ 3 × 5 – 5 = 0

To sum up, the stark differences between the school way and the mathematical way as evident in the simplification of the given expression are listed as under:

1. 3 in '6 × 2 ÷ 4 ÷ 3 × 5 – 5' is '3 units' (i.e., 3 ones) in the school way, and it's '3 fours' mathematically.
2. 5 as the subtrahend in '6 × 2 ÷ 4 ÷ 3 × 5 – 5' is 5 roses in the school way, and it's '5 dozen roses' mathematically.

Importantly, the school education system has not formally explained ANY of the hundreds of logical explanations missing in 'math education' in the past 200 years. Math has been taught only for encouraging number-crunching abilities, and it works only for a few.

School math, in its current form, must be reformed outright.

‘Cent Percent Mathematics’
A language like no other

Significant laissez-faire abounds in mathematics education to this day. It has been a case of labour without an ideal, and goal; students are mostly faceless in K-12 but most so in mathematics. An interesting aspect of mathematics education is the extent of similarity in the pedagogical approach, planning and resources across nations; the level of abstraction is alarming and universal. Complete disregard for the socio-cultural contexts of the learners is common. There is more, it has lived with opposite posturing of leading universities of the world, for instance, Cambridge University has an academic stance that is mathematised, but Oxford University plays that down (and this does not mean any less for either, just that both are feasible stances).

K-12 mathematics education content and processes have undergone several reviews and reforms in various parts of the world yet student performance in mathematics continues to be dismal.

Children and mathematics

Childhood, school years, and mathematics bear an unholy relationship. Extended childhood, i.e., biological childhood (up to around 7 years of age), juvenility, and adolescence are almost entirely spent in schools. Expectedly, the importance of the quality of school years cannot be over-emphasised – extended childhood years are important formative years. They shape most of who children become post-school and the rest of their lives.

However, the biological gift of extended childhood being the best learnable years needs to be complemented by ensuring that every child is (truly) happy. Only a happy child can learn. Happiness for a child is most driven by a continued sense of 'free play', in more specific terms – control over the pace, and steps to the outcomes (adults also share this need). Getting appreciated for the outcomes is a way of feeling control (appreciation of others is central for children, far more as compared to adults). Happiness increases engagement in activities/tasks, and engagement is key to learning.

School math is a spoiler of happiness, and challenges due to school math starts in preschool itself for some children. In upper-primary grades, the majority of the children are hit by anxiety due to math and by the time children reach middle school, fractions and algebra break the hearts of most children.

Math is the most important reason why schools have ‘failed’ a big majority of all students in the past 200 years (eventually, even if not literally). However, mathematics is a meta-subject in the sense that success in math is socially rewarding in all cultures, and the unfair poor scholastic achievement in mathematics hurts self-esteem and optimistic worldview in the childhood, teens, and young adulthood years.

In many societies, definitely in all the Asian ones, getting math right is crucial, so a lot of play and free time are invested in supplementary math education. Most children can’t cope with this obsession with mathematics, especially when no adult in their lives – teachers, tutors, parents, siblings, others – can do one bit of help in getting a hold over mathematics. Worse, the overwhelming majority of children come to realise this stark order of life in their primary school years.

This math-induced stress spills over to the general psyche and over time induces a lack of confidence and deteriorates performance and interest in other subjects. Sooner than later, childhood turns into a burden to be done away with as soon as possible. Indeed, many Asian societies look at childhood as just a passing phase to adulthood and ignore the emotional well-being of the children.

The lack of poor application of child psychology in school education is writ large all around us, for example, the 'happiness curriculum' being deployed in schools run by the Government of the National Capital Territory of Delhi, focuses on 'overall' development at the cost of academic excellence, overemphasis on hands-on but minds-off 'play-way' methods. Being happy is a state of mind, a worldview of the state of things in life, not episodic 'happy experiences/events'; only happy adults around children make a (truly) happy child.

What makes math the easiest language to learn?

Math is much easier to learn than natural languages and the 'genetic languages' such as art, music, play, gestures, etc. Interestingly, there are quite a few reasons why math is much easier to learn (when analysed for school-level math).

1. There are just 10 digits (alphabets) in math (0–9).
2. Math is a far more compact language, e.g., the following sentence, in English language sentence – I had 8 apples and I gave one apple each to two friends – is mathematically
   8 apples – 1 apple – 1 apple, or 8 – 1 – 1.
3. The grammar ('rules') of math is limited. It includes the 4 arithmetic operations, equations, vectors, etc., and these rules are also tightly linked together (thus, the volume of basic rules is even less).
4. There are a few exceptions and conventions in the grammar of math. An example of a convention is – the axes in a graph are drawn at 90° to each other, and the x-axis is horizontal, with numbers written from left to right.
5. Math's grammar is universal (2 cats + 2 cats = 4 cats, for all of humanity) – math is the only universal language.

6. It is much easier to continue to understand newer concepts in math. Concepts in math are linked in a rigid hierarchy – thus, every new concept is based on other definite concepts (that we can easily identify and specifically learn). For instance, if we comprehensively understand the addition of 1-digit numbers, we can add larger and/or multiple numbers.
7. Practising math can be a self-paced solitary activity that may not require the company of others (unlike when practising natural languages).
8. Math is much easier to explain to others. There are numerous alternate ways of demonstrating and visualising every concept (because math is as 'real world' as it can get).

It is easy to jumpstart a new journey of discovering math all over again!

Math can easily be the second language at home, and that would change everything for us.

Embracing a new language

The most important feature of natural languages is that they grow on you with daily usage; we keep acquiring new nouns, verbs, adjectives, adverbs, and phrases without any specific effort. And the languages we are acquainted with are not forgotten even when we do not converse in them for a long time. When we use mathematics as a language, the layers of mathematics that we know must grow as we deal with quantities in our regular day and it too would stay with us for a long term.

On the contrary, unfortunately, math concepts fade away from our conscious memory sooner, even for the best educated ones. School math textbooks are responsible for the current state of math education on three grounds – educationally unsound partitioning of mathematical content strictly along grades, extreme teacher-centricity (designed as a teaching aid, rather than a learning aid), and far removed from real-world contexts. New-genre textbooks are needed to be learner-centred (comprehensively useful for independent learning), and free of distortions due to grade-caused syllabi, and abstraction.

Briefly, mathematics as a language implies 'reading' a real-life situation as '3 trays of 4 cups each', then writing it mathematically as '3 × 4 cups'.

Improving the content of thinking

One way to better contemplate any situation is to pour in more relevant and appropriate knowledge of the various dimensions of the situation. For example, while discussing the baking of a cake, the more we know about the physics and chemistry of grinding, refinement of flour, heat, blending, baking powder, water content, ratio of ingredients, etc., the better would be the outcomes of the baking process.

Clearly, this mode of better thinking requires extensive knowledge acquisition, sound foundations of as many domains of knowledge as possible, experimentation (an independent mode of knowledge acquisition), sharper and more vigilant observational skills, better command over the language of reading/academics, etc. It is a demanding way of training to think better.

Who doesn't need to think better?

The world would have already been a wildly different place if 200 years of the current school education system had gotten mathematics education to work for every student.

A robust math education dramatically trains us to think better – think in a more reasoned, logical, ordered (systematic), and in a more confident way. Nothing trains us to naturally think about the 'why' of everything as does good math education. Persistent and organically connected 'whys' have a distinguishing destination – right thinking (reasoning). It is only apt to quote Jordan S Ellenberg, a mathematician who is also a fiction and non-fiction author, '*Math trained habits of thought … how not to be Wrong.*'

Improving process of thinking

The other way to think well is to hone a better way of thinking 'anything', habits of mind as a default thinking cap. The science and practice of decision-making – a specific, goal-driven thinking process – are well-researched and documented. There are several very popular books on the subject, such as *'Six Thinking Hats'*. However, math catalyses a more rigorous ability to think, and just a good understanding of math is required to use it for better thinking.

Learning math automatically nurtures a positive experience by asking 'why' for everything. Math is purely logical, and almost everything is explainable in terms of universal reasoning (and some things which are not explainable are universally agreed conventions, for example, composing numbers from the right).

A good education in math instils the natural habit of thinking, reasoning and seeking a deeper understanding of all numerically expressible situations. It helps us translate real-world situations/relationships/events into mathematical expressions (models) on paper!

Needless to say, school math strips math education of all shades of mathematical thinking. Mathematical thinking is not about arithmetic or algebraic equations, it is about training the mind to think more logically, reason better, and see patterns in the behaviours of numerically definable situations/events/relationships.

Of the two – improving the content of thinking and improving the process of thinking – the latter takes far less effort to acquire, and it is a very generic ability.

Mathematics is moral (too)

Morality and mathematics in this context are about the educational value of mathematical concepts and objects in consolidating certain moral values among children. A very interesting bit about mathematics is that for children the 'repetitiveness and rigidity' of mathematical thinking, and even the 'methods' are moralistic.

Young children have little understanding of the meaning and demands of moral behaviour. They tend to think of it as a series of rules for what is right and what is wrong, and in a fairly literal way. Mathematics may actually come across to them as the most 'moral subject' from this perspective.

Many foundational number, arithmetical, and algebraic concepts are quite coloured in moral stance, mitigating moral dilemma. Here is just a sample of a few concepts promoting moral worldview and outcomes.

Fraction may be mentioned as the first evidence of moral fibres in the otherwise rational roots of the concept which is based on the concept of exactly the same parts of given things. For instance, visualise a birthday party at home with a circular cake and 16 friends. The cake would be evenly split into 16 pieces, and each person is expected to pick a piece each, taking exactly $\frac{1}{16}$ part of the cake. Thus, all get/take exactly equal sized slices of the cake.

Rational numbers stand for legitimacy – an essential base for higher moral standing. It is a typical mathematical character to be upright, and proper. The acknowledgement and formalisation of improper fractions, together with proper fractions, as distinctive numbers is very instructive, and in the true mathematical sense teaches us to be inclusive.

Irrational numbers remind us to live by the standards. Once rational numbers were accepted as a set of numbers that represent our world for the most part, and central to most mathematical expressions and computations, other numbers were named to be the opposite, despite a world of infinite irrational numbers.

Permutation reduces the chances of loss of face and at times may cause ethical conundrum. The mathematical formula for the number of permutations is magical, reality reduced to a number in the quickest time possible. Short of the formula, one would not easily know if one has exhausted the possible distinct arrangements of the given set of objects. The risk of missing out on an important or critical arrangement could be embarrassing, for instance, the lack of cognizance of a certain order of precedence of dignitaries that is unacceptable could cause ethical challenges. Permutation comes in very handy in capturing and eliminating all possible moral discomfort.

Combination plays a similar role by offering the arrangements that portray equality of all participants, or target objects. The combination is a particularly helpful construct in avoiding moral lapses.

Algebra brings in the sense of transparency and truthfulness – two important moral strands at a societal level. Algebra rests on the public and precise quantification of relationships. Expressed as equalities, or inequalities algebra promotes appreciation of the connectedness of things.

Probability considerations and computations are morally influenced. Estimation is an integral part of the idea of probability. All estimates are also affected by the values and beliefs of the people involved, personal gains, losses, or social motives must be kept at bay in probability, qualitative as well as quantitative. Mathematics trains us to attain a higher moral state.

Data analysis is now the bedrock of social sciences. Not only are statistical tools analysed for us as inputs, but these tools also act as ends in their own right. The world of Big Data has little in overlap with our current one – there is so much of data, new kinds and endlessly varied – that mathematics is the only reader and translator. Social researchers and commentators cannot confidentially critic without a mathematical sense.

More examples of mathematical concepts and thinking being a partner in our moral compass may not be necessary for the limited purposes of this book.

One essential goal of specifically listing how mathematical concepts carry moral import was to bring more spotlight on the connection between rational and moral. First, morally right could well be rational too. Second, morally wrong may also be rational, such as when self-interest prevails. Third, not rational may be moral, but it better be rationalised to the extent possible, for example, it is morally better to offer more share of food on the table to expectant mothers, but it would help if there is some rationality behind 'how much' more (and mathematics can help). Fourth, not rational and not moral is not about mathematics.

Math is the fairest of all

Math is the best language to conceive, communicate and act fair. The best defence of fairness of an act, or decision, is by mathematically presenting the same, and this is ancient wisdom. For instance, in the fourth century BCE, in the set of (twelve) books, *Laws*, Plato talks about the construction plans and settlement of people in a new city, called *Magnesia*.

Each household was to get an agricultural plot and a dwelling plot, in the most equitable way possible with the fairest distribution of the productive asset, the agricultural land. The dwelling units were to be of the same size and the households were supposed to be capped at the same number of members.

The book proposed that the size of the agricultural plot must reflect the yield of the land; for example, bigger plots to be carved out for the less fertile land areas. Mathematics was to be at the heart of ensuring equity in income from agriculture. It is indeed surprising that the idea of fairness is evident to even two-year olds, and they understand it best when it is mathematically substantiated. For instance, a toddler easily understands that a smaller share of a sweet for her is fairer because she has a smaller tummy compared to the older sibling.

Math is the most convenient language

‘*Laws*’ also mention *Magnesia’s* number of households to remain frozen at 5,040. Plato defends the choice of 5,040 by stressing that it is a very ‘convenient quantity’ – for instance, 5,040 houses, or agricultural plots, could be homogenously grouped together in many ways. 5,040 has over 50 divisors! And that means the houses and agricultural plots can be blocked together, at multiple levels, in ways far more than one can imagine, or needs to.

Let us say, all of the 5,040 houses could be organised into 6 houses on each street, 5 streets to make one sector, 7 sectors to make one colony, 12 colonies to make on administrative unit, and thus, the 5,040 houses to be split into 2 administrative units. Many such alternative spatial planning is possible with the easily divisible number 5,040.

Math sense is common sense – living, earthy, personal

Common sense is the body of knowledge that we all have simply pushed out of everyday life and activities. It is largely created out of our private inferences, observations, explanations, and visualisation of our world. It is what makes the navigation of everyday living fast, efficient, and effective at the personal level. It also helps us be unaffected, in both good and bad ways. Interestingly, it is not a well-organised body of knowledge. Empirical testing is not possible for what is commonsensical, it is unexamined.

A really happy aspect of common sense is that it is always there in all of us, consciously and subconsciously, and in unique configurations. Math sense is a part of the common sense we all live with. The foundational ideas/concepts of math are commonsensical, they are all innately present and continually developing as we go through the routine of life and work.

Moreover, the bulk of K-12 math is essentially foundational along the different ‘branches’ of math. It is indeed humbling and invigorating to realise how the appreciation, understanding, application and 'mathematical modelling/expression' of all 'branches' of math, such as numbers system, arithmetic, algebra, statistics, geometry, calculus, etc. may be the most common denominator across humanity, the most typical characteristic of all humans. Of course, this is expected because rationality, or reasoning is the most universal of human traits.

Pure logic; rigidly hierarchical reasoning typifies math. Understanding the basics of math, in all its multifaceted expanse, just needs the 'presence of mind', a thinking person. Let us explore some examples of how mathematical sense is quite commonplace. Educationally, it implies that there is nothing to teach/introduce in the basics of math.

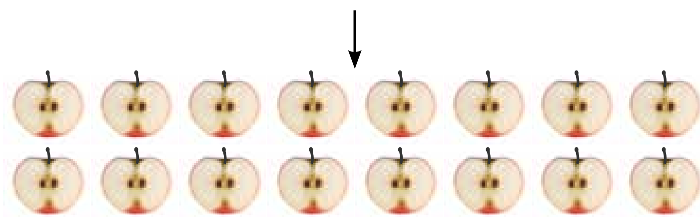
We all commonly organise a wad of different denominations' notes by the likeness of the notes, all the notes of a denomination are stacked consecutively, together. Organised notes make it easier, error-free, and quicker when making payments, or computing the total value of the cash in the wad.

This is an expression of the fact that we all know perhaps the most important mathematical foundation – numbers/quantities are made of different packets of quantities. For example, 345 is 3 packets of 100s, 4 packets of 10s, and 5 packets of 1s, just as ₹345 will be arranged as the 3 hundred notes together, 4 ten notes together, and 5 one notes together (assuming all possible sets of 10 notes of ten are exchanged to 1 hundred, and all possible sets of 10 notes of one are exchanged to 1 ten).

Another profound foundation of math is children learn addition, for example, how '3 + 5 = 8'. No one has to know addition to tell how so, they simply visualise 3 things and 5 (similar) things and find the total count of it to be 8 things. Indeed, the only way to know the sum of '3 + 5' is to count the final quantity, and we all know that; school math takes the abstract route of memorising, methodising '3 + 5'.

The idea of operations is in us already. Let us evaluate if obtaining the quantity represented by $8\frac{1}{2}$ is intuitive and that the idea of division does not require any teaching/inputs. *There is no special (mathematical) knowledge needed to find the quantity of half apples in eight full apples* (except of course the number system being used to express the quantity). It is visualisable by one and all that there be sixteen half apples would be obtained from eight apples; it is as easy as imagining each of the eight apples being cut into half and that count of half apples being sixteen. Pictorially,

8 apples

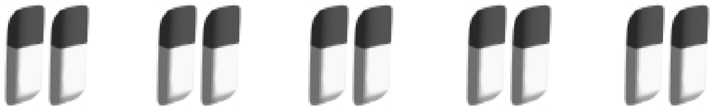


16 half apples

Half apples in 8 apples

The idea of the operation is so sharp in us all that the difference between 2 × 5, and 5 × 2 is almost missing in school, globally (both are taken to be just 10, totally similar mathematical statements). However, ask anyone if the following two situations are similar, and the resounding response will be 'No'. The following pictures make it evident:

2 sets of 5 erasers (2 × 5)



5 sets of 2 erasers (5 × 2)

2 × 5 and 5 × 2 represent two different realities

Similarly, it is commonsensical to know the price of 2 kg of potatoes if the price of 1 kg is ₹50. Ask anyone the price of 1 kg of apples, if ₹110 was paid for $\frac{1}{2}$ kg of apples and the carry bag (at ₹10 per piece). Expect the correct response from all. And it is not easy for many to solve the algebraic expression representing the situation: Find the value of x in '$\frac{1}{2}x + 10 = 110$', if x is the price of 1 kg of apples!

The idea of algebra is universal! The price of 16 kg of potatoes, when the price of 1 kg is known, becomes difficult to obtain, but that is only because of computational challenges, not the idea behind the computation. And there is more – everyone 'sees' the math in the aforementioned example as they solve '$\frac{1}{2}x + 10 = 110$', as under:

In most minds, the first step in 'solving' the above is to 'isolate' the apples by removing the carry bag, and reducing the carry bag price from the total price (subtract 10 from both sides. The right way to see the operation, not to see it as taking the 10 to the other side and subtracting it from 110 as school math commonly teaches). Then the price of $\frac{1}{2}$ kg becomes the same as the price paid, i.e., $\frac{1}{2}$ kg = ₹100.

The challenging idea of percentage is commonsensical too. The percentage is a relationship of quantities, it defines a relationship in terms of 100, for example, if for every 100, the relevant quantity is 50 then we call such a relationship as 50 per cent. Then, for every 1000 we would have 500 (500 is 50 per cent of 1000).

Commonsensically, all of us know that if for every 1 we have 2, then for 50 we have 100; if for every 1 we get $\frac{1}{2}$ then for every 100, we get 50.

$$80\% \text{ of } 200 = 80 \times \frac{200}{100} = 160$$

All the basic idea of geometry is innate, the idea of space is a basic intelligence. Everyone understands the area as the amount of surface, people know more paint is needed for a bigger wall; the idea of perimeter is nothing new, people know that the cost of fencing a space is related to its length and breadth; seeing a volume as a sum of sub-volumes is natural, and so on. Ironically, 'something' goes very wrong in school math that area and perimeter questions are mostly wrongly computed by children.

The idea of pictorial representation of quantities is nothing new to anyone, 'graphs and charts' are commonsensical. It is the knowledge and importance of 2 and 3-dimensional coordinate points that are not lost on anyone. For example, people think before they place a source of light in a room to best light up a room; people can read and make sense of maps without any real training. The idea of pictographs is as natural as differentiating between the quantities of two heaps of the same things.

The fundamental idea of average (of data), and the appreciation of different kinds of averages is surprisingly common. People know what they mean and communicate when they say that members of a particular family are tall, they do not imply that all members are similarly tall, it is more to mean most of the family members are tall; when talking about the popular colours of tops/shirts, people actually use the idea of mode to declare that, and similarly, when people talk of the likely duration of travel to reach a place, they actually share the modal travel time (the most common travel time in the past).

It may not surprise now that the idea of probability is also widespread. It does not need any example to substantiate. However, for the record here it may be recalled that we all meaningfully talk of the likelihood of specific events happening or not, and those decisions do involve computations similar to the way we mathematically compute probability, e.g., when predicting rains following windy and cloudy conditions we actually 'quantity' how many times it rained in such conditions, and when all it did not.

The essential ideas of calculus are also for all to see. We see three ideas, among a few more, to be central to appreciating calculus is fairly commonsensical – the importance of instantaneous values of changing quantities (speed, for instance), some quantities are derived out of others (they are not measurable as a standalone quantity, such as speed is derived out of distance, and time), and the idea of integration as sum of similar parts (zig-zag parts versus standard-geometric parts of bigger area and volume).

It is universally known that in an accident the speed of the car at the instance of the collision is all that matters, and not how fast or slow it was before that instant. The idea of derivative is similar to the commonplace understanding of pressure, a quantity derived out of force. The idea of anti-derivative is easy to observe around if we realise that it is more deterministic to have standard geometric shaped things than the slightest 'zig-zag'/irregularly shaped things in predicting (surface) area or volume of things.

Mathematics education must always build on the powerful and beautiful mathematical mind that we all are born with. And then no one will ever be left behind in excelling in math, forget passing the grade mathematics.

School mathematics kills common sense

Abstraction is the opposite of common sense, the latter is earthy, rooted in real-world situations and their quantitative expression.

School math first decapitates thinking and visualising and then in later years goes back to it via word problems. But by then word problems cease to be easy.

Unless we learn to abstract from real life situations right from the very beginning, even 3 + 5 = 8 is no debate in an abstracted form where we assume the units of 3 and 5 to be the same, but this is a big assumption.

School math is highly abstracted, for instance, in the kindergarten itself numerals are equated with numbers, and the meaningful distinction between the two is snubbed. Thus, 2 + 4 = 6, always. However, in the world we live 2 + 4 is mostly not 6.

Common sense refines with time as we 'get wiser' with responding to changing situations, and conditions. Common sense refinement cannot happen through instructions/teaching. In fact, any teaching/instruction will deconstruct and interfere with the refinement of math sense. Common sense is personal and grows out of 'personal lessons' not universal lessons. Common sense is not imparted, it 'incrementally, discretely develops' at undefinable and unexpected moments and situations.

Math is the language of artificial intelligence

Mathematics is the language and substrate for artificial intelligence (AI); the atom of machine intelligence is data and molecules, the algorithms. No aspect of life and work is untouched by AI applications, and every adult and child must understand and appreciate the data and algorithms behind the specific AI applications.

Is mathematics the ONLY hope for humanity?

One of the more simplified statements of the developmental condition is that the quantity and quality of development of a community are directly proportional to people's proficiency in the language of academics (the language of science and technology). Natural resources, climatic conditions, etc., are secondary.

However, cultivating the highest competencies in natural languages (i.e., language of academics) at a mass scale is a science we still know little about, except perhaps the role of extensive reading and conversations in language development. But learning to read is a hard-earned acquisition, and quality conversation is a lost art. Does it mean a dead-end for us?

Fortunately, not. Mathematics as a language can do the trick. Mathematics is already a global language (the only one), and every human can master it. With artificial intelligence (AI) becoming the new global infrastructure, mathematics is increasingly becoming the language of all knowledge. It is projected that mathematics will be to biology, by the 2050s, what it is for physics today. And the unimaginable part is that the mathematics required for AI is the current K–12 school-level math syllabus!

The only pre-requisite for learning math – natural language

Surprise it may, if only we can precisely express a situation in a natural language, we can easily translate it into a mathematical expression.

Talking of learning languages, we learn a language on the back of another we already know. A second language is learnt on the back of parallels with the first, and a third on the back of its similarity with the first and the second. And how do we begin the journey of learning the very first natural language? The language of art, music, gestures, etc. is a gift to all children, we use these for learning the first natural language. For example, we learn to differentiate between the cups and glasses, how-so-ever imaginatively crafted, by visualising the cups and glasses already seen physically, or in pictures.

Section III

The AI-age Mathematics

Obvious as it may be, mathematised mathematics has to be the ‘real mathematics’, mathematics as the language for all quantified objects and contexts. To us, the authors, ‘Cent Percent Mathematics’ sounds more apt as the name for real mathematics.

Epically, ‘Cent Percent Mathematics’ is just the mathematics that we all need to learn and hone in the Big Data, intelligent times. A mathematical mind that can mathematise real-world situations is all that we need, computational skills mean little.

Soul-searching Mathematics

The first step in a journey that will take one the farthest is the shortest – it is standing still to soul-search, to think through ab initio. A dimension of soul-searching is evaluating (relevant) history to know all about the present.

Understanding mathematics would be much easier and better if we access its foundations historically. This book is no place to discuss why mathematicians must also be long on mathematical history except that it would encourage and empower creative and flexible thinking, besides massively expanding the repertoire of foundational knowledge. It would also make mathematics knowledge more soulful, and, thus, more enjoyable, and unhooked.

Augustus De Morgan, a famous logician, was very vocal for his emphasis on the critical relevance of the idiosyncrasies of processes of mathematical discovery and famously equated them to the intellectual sharpness of ex post facto mathematical propositions. To him, all mathematical research would benefit from the world of such insights. He lamented, *"It is astonishing how strangely mathematicians talk of the Mathematics because they do not know the history of their subject. By asserting what they conceive to be facts they distort its history. There is in the idea of everyone some particular sequence of propositions, which he has in his own mind, and he imagines that that sequence exists in history; that his own order is the historical order in which the propositions have been successively evolved."*

Galileo has been called the father of observational astronomy, the father of modern physics, and even the father of science. He is also credited with the 'mathematisation' of science. He wrote that the Book of Nature is 'written in mathematical language, and its characters are triangles, circles and other geometric figures, without which it is impossible to humanly understand a word. Without these, one is wandering in a dark labyrinth.' We need to go back to some basics.

Why is math important?

Math has always been an important language. People who excelled in math also excelled in science and technology and had good chances of professional success. This does not at all imply that math is a pre-requisite for professional success, but, all the professionally successful people without a happy relationship with math did have to depend on some math-affiliated professionals to make better sense of their wealth, and grow it financially.

However, the importance of math has only multiplied in the last few decades. The advent of computing power and digital communication started to push math to the centre stage of technology and everyday life. Computers and the internet are feasible due to complex math (and simple math too). A computer is, simply put, a mathematical product. For example, everything, including words such as 'great', special symbols such as '@', and numbers such as '12345', is recognised and processed by computer using 'unique binary number codes' for every word, symbol, and number. Even photos are turned into very large binary codes.

In 2020, there is almost nothing in our lives that is unaffected by information technology. We can well imagine what would be the impact of information technology in the 2030s and beyond, when today's adolescents become adults.

The fast-penetrating Artificial Intelligence is literally founded on mathematical modelling of real-life situations/conditions that uses linear algebra and introductory calculus. To be creative and successful in the massively digital world around us, the ability to naturally think mathematically is imperative.

For example, each piece of art, design, or music is increasingly becoming a unique set of binary number codes. In fact, this is how Facebook recognizes our photos, and even automatically tags us on our childhood photos as each face is a vast, unique set of binary codes (that reflects the minutest features of our faces).

3D printing, a big source of revolution in our lives, takes data (numbers) as input (besides materials) to print whatever we seek. For example, to print/make a 1-litre plastic water bottle using a 3D printer, we need to feed the bottle design and specifications in the form of a big binary number code, not as a physical mould of the bottle to be made.

Friendship with math pays

For school-going children, there are literally no model philomath adults. Fear of math is probably the most common social affliction. This creates a social context for children which is not conducive to developing competence and interest in math. Math needs to be learned again by all parents for their children to learn math.

A happy association with math is almost a pre-requisite for a happy childhood and a happy school-life. A disproportionate amount of time, attention, and money is invested in supplementing and complementing math education at school, and this affects performance in other curricular and co-curricular subjects and interests. Struggling in math has a cascading effect on overall learning. Thus, math must be specially attended to.

Evidently, for the parents of school-going children, the easiest way to gift them a cherishable childhood is by becoming a co-learner in math and their preferred math buddy. Parents do not need to be ahead of their children, they just need to be with them as they discover math concepts. Learning math to 'teach' math to your children will take the least effort compared to all other subjects/ domains of knowledge.

This implication for parents is important irrespective of schools and teachers doing a better job of teaching math.

Biology and math together?

To top it all, in this century, we expect to mathematise just about everything we know of today. For example, by 2050, math is expected to be to biology what math is to physics today (physics is almost mathematics). By 2050, one can expect the human body to be a set of deterministic 'mathematical equations' with individual differences being reflected in the data inputs to the equations.

A significant portion of genetics depends on quantitative data. Many mathematical techniques (such as probability and standard deviation) are used extensively when studying genes. We use probability to predict how frequently a particular genetic configuration manifests. Emerging fields of study such as bioinformatics use statistics-based computer programs to scan DNA and analyse genes.

As commonplace evidence of math getting simpler are the claims regarding education to become data scientists. The advertisements talk of the possibility of becoming a data scientist without the need for a specific technical background (i.e., without much of a math background)!

To sum it up, math is becoming more important by the day for two important reasons – first, math is the language of all technologies and we are witnessing an explosion of new technologies, and second, the 'must-know' math is made easier (the 'cent percent' mathematics).

Why math is the best fit to ensure humanization?

Mathematics is truly universal in another way also – no society or country may know, or harness any additional, newer mathematical knowledge, that remains hidden from the rest for long. This is very unlike scientific knowledge where differences among societies and countries are the norm and the source of different development trajectories. Mathematics seamlessly and equally belongs to all of humanity, it takes nothing to access, apply, and augment mathematical knowledge. In fact, Srinivas Ramanujan's recognition in the world of mathematics gratifyingly displays the distinctive qualities of the architecture of mathematics – integrity, verifiability, transparency, infinite jig-saw and sheer elegance/joy.

It is almost a norm for individuals, communities, regions, societies, and nations to be variedly placed when it comes to science and technology. Mathematics is a great equalizer, it is a sphere of knowledge that is peerlessly zero-cost to research and apply. Besides, all of K-12 mathematics education could be comprehensively contextualized for every kind of society that exists today, without any conceptual dilution. The first universal, default human characteristics can only be accomplished by mathematical thinking, whenever and however it happens.

History is replete with how civilizational stature came on to societies that placed mathematical thinking on a higher pedestal. We can well imagine the metamorphosis upon expanding mathematical imprint to all of humanity.

Mathematics is (just) the foundation of 'Being Human'

Mathematics is a class of its own for another reason – it consumes a disproportionate amount of time, effort, attention, resources, and prayers of families and students when compared with all other subjects put together. And this is true of all societies, communities, and families. Naturally, success in mathematics in K-12 leads to freed-up time and resources for due attention to other subjects, and that success directly supports better learning of science and economics.

Ensuring every child succeeds in mathematics is to put the children on the path to becoming multi-intelligent adults. On another plane, a sharp focus on mathematisation imparts automatic momentum to the all-round development of an increasingly larger percentage of children.

Indeed, it would be just and true to assert that the role of mathematics in humanisation is as informal, indirect, and multifaceted, as stark as it is in changing habits of thinking.

Mathematisation = Happy childhood = Successful adult

The effective idols of K-12 are the 'mathematics geeks, toppers', notwithstanding their performance in languages, arts, social sciences, and even science. The life of mathematics toppers in school years is enviously sorted and ensures coveted perks. Mathematisation will ensure a growing fraction of children live developmentally appropriate childhood years and happier families. And happy childhood nurtures confident, caring, and learning adults; only a happy child can ever learn or seek to learn.

End of confusion – mathematics as a language

A raging debate, rather confusion and conundrum, in mathematics education, since antiquity, is how the idea of language literacy has been borrowed/brought into mathematics education. Quantitative literacy as the goal of K-12 mathematics curricula has reduced mathematics to arithmetisation of the curricula – absolute abstraction (even simplest 'word problems' turn nightmarish for children, globally), Σ methods, and 'step by step evaluation'. Thus, courses in algebra, geometry, data, vectors, trigonometry, and even calculus are have soul sucked out of them, and turned into mechanical chores.

The quantitative literacy and mathematics disjoint must end, now, and forever. There is only mathematics to be learnt, the real (100%, full) mathematics. Of course, real mathematics is 'mathematics as a language' and it is constitutively quantitative literacy in spirit, conceptually threaded and networked in mind, and methodised in body. Once we start learning mathematics as a language, we will be perceiving, thinking, and expressing situations and objects quantitatively, and more accurately.

It may be interesting to relate mathematics as a language to the most popular sports in the world – soccer and cricket. In both games, the meticulous position of players in the field is more than half the defence and attack. This positioning is almost entirely mathematical – visualising the speed of the ball, the speed of the players chasing or approaching the ball, the distances and the angles between the positions, the shortest distance between certain manned and unmanned positions, and more. And this spatial intelligence is visible in the winning captains.

The most important triumvirate

Nature Maybe, the weakest link in the institutionalized school education system is the 'know-why, know-how, and know-what' of nature. At best, we study nature in fractional pieces and all mostly insularly, for instance, the water cycle is studied as a physical phenomenon, as an example of evaporation and condensation but we miss the essence – evaporation and condensation as a 'natural phenomenon', as a routine yet extremely sophisticated and 'organic' element of a whole. We completely miss to appreciating the beauty, and intricate details of the water cycle.

Nature may best be seen as the whole, the one that holds all the living and non-living things in the world that are not created by humans, as well as all the events and processes that are affected by humans. Nature follows an intricate web of cause and effect, 'rules' that are incessantly, and unchangingly at display. Nature lives by an unwritten code.

An interesting part of the aforementioned 'definition' of nature is that acts of all living beings, except humans are an integral part of nature; we think and behave in 'unnatural' ways, such as upsetting the food web of a region or planting a non-native life in the region (that may hurt the region in the long run).

Nature and science Science is the name given to the body of knowledge that is essentially a collection of all the codes of nature's behaviour that we have successfully decoded and articulated mathematically (such as in physics and chemistry), or descriptively (as in biology). We express such knowledge as scientific laws, and the laws are true in all conditions as defined in the letters of the law.

The collected laws in science cannot exceed what actually happens, or does not, in nature; a confident understanding of the 'nature of nature' is the best science can get to. For example, science is still short on fully understanding the ways of nature, such as weather. This is best evidenced in the quality of predictions of weather (science) models for shorter time periods – Nowcast (24- hour weather prediction) is less accurate than short-range forecast (1–3 days).

We discover, not invent (make up) scientific facts and laws. We observe, experience, or analyse acts in nature, and then create statements about a particular act in nature (called hypothesis). We then set up experiments to verify hypotheses, the true ones become laws.

Science and technology The relationship between nature and science is repeated in science and technology – technology cannot exceed the body of scientific laws at any point in time. Technology is 'man-made nature', replicating acts in nature. The implosion of the Titan at 30,000 below sea level is an example of all the technologies used in making the submersible live up to all the science we know about such conditions.

Similarly, on the hardware side, quantum computers are waiting for manufacturing 'near zero-defect', extremely 'quiet materials' materials, not just the base material but also at the surface and at the interface of materials. We know the kind of material, and the science, but not enough to make them in quantity.

Nature of technology Now the crux of this discussion – technology is 100% mathematised knowledge in action. Every technology, the simplest ones, is 100% predictable in its input-output transformations. The 'science' of technology implies the same behaviour, time and again; for example, we would seek the repair of a faucet we are used to if there is unexpected variation in its dispensing for a given turn. An e-car will run the same distance at full charge under exactly similar driver, traffic, weather, and road conditions.

Mathematics is at the heart of the servile behaviour of technologies.

Humanity and technology Science is human's earliest and most passionate pursuit for the sake of survival and a better life. Naturally, technology is human's earliest friend and the only presence in our lives that has continuously stayed and propped the rise as well as fall of civilisations. Intensification of technology is a secular trend.

Yet, something has dramatically changed about technology, it is not just changing society, it is changing what it means to be human. Its pace, sweep, acuteness, unforeseeable innovations, and profoundness are all new. The best of all, in the end, technology will propel us to what is called by some as Society 5.0 – a humanity that best balances real and virtual, sustainably.

In that process, humanity is getting unprecedentedly mathematised.

We and humanity Humanity is a network of innumerable, overlapping societies. Yet, society is humanity's womb. Society is the magic in our lives. Society is the invisible fabric that holds us all together, much like the way gravity is like an 'endless taut fabric across the universe', a natural outcome of the existence of celestial bodies in space. It is that 'fabric' that warps, bends, pulls, or pushes everybody, under the 'weight' of the other bodies around them; that is also how heavier bodies exert more force on others, more gravitational force on others. Gravity holds the universe together, society holds the life in the universe! Yet, the new millennium is not all good news for societies. Despite the inherent robustness and vibrancy in the very design of societies, we are steadily becoming 'thin cultured', hollowing out societies. But we must not forget that the need for society is biological, much like oxygen.

Every one of us must place society at the centre of our economic and political life.

The irony of it all

Mathematics education is fractured in a very paradoxical way – the real mathematical foundation and applications for both ends of school students are actually the same poor quality. Well over three-fourths of all children are formally evaluated to be strugglers in mathematics@school. The minority of students evaluated to be good, or outstanding in mathematics@school are just that – whatever schools call as mathematics; they have no way to know better – anything beyond the totally methodized, blindly practised and abstract number crunching. Here are ten real mathematics questions from K-10 syllabi, and see for yourself how many of any of these ring a bell –

- How $\frac{1}{0}$ is undefined, but $\frac{0}{1}$ is 0?

  Hint: The definition of division.
- II + IV = VI? Yes/ No/ Depends
- '7 – 2 = 5' is 'uncertain'. True/ False/ Depends
- Fill in the blank spaces to make the statement on face value and place value a correct one.
  Value of a ……. at a place in a …….. = Face value of the digit × ….. of the ….. at which the digit is placed
- What is 2 × 3? 6/ 3 × 2/ both 6 and 3 × 2/ neither 6 nor 3 × 2
- Visualise $\frac{3}{4}$ as a division and $\frac{3}{4}$ as a fraction.
- Compare $\frac{8}{\frac{1}{2}}$ and $\frac{8}{\left(\frac{1}{2}\right)}$.
- Tan^{-1} gives us an angle. True, False, or it is a bit ambiguous question.
- Why 'combination lock' is a misnomer, mathematically incorrect?
- Derive the mean formula in 3 steps. Recall, the formula is mean = sum of numbers/the total count of numbers

Of course, we, the authors, would only be delighted if the questions seemed familiar. Answers are never important. We have not attempted but would be happy if you seek artificial intelligence (AI) chatbot ChatGPT's help with these questions.

Their quantitative literacy and confidence are severely hurt, bordering on paranoia but the remaining minority are stressed over mathematics@school.

Educational transformations need epochal thrust

J. David Markham, an internationally acclaimed historian, emphasizes the organic connection between education systems and the state of societies, stating that much of the history of Europe can be observed in the ascent and decline of its educational systems.

We must be real. Can populist democracies or the incumbent elites (who are from current institutions) reform education and seek the same outcome K-12 for all the children of the world? History educates us that this is next to impossible, history helping us with lessons for a better present.

As evidence, three epochal moments are particularly valuable lessons – the return of the ancient Roman Empire as the Holy Roman Empire, Islam's Golden Age that collated and added to the knowledge of the world that also triggered the Renaissance in Europe, and the French revolution aimed to establish lofty idealistic goals of liberty, equality, and fraternity. The French also revealed that the first (clergy) and the second estate (aristocracy) sought education reforms. The poor and the peasants (the third estate) did not seek educational reforms, it was the.

Markham particularly emphasises the connection when he links the fall of Rome (Western Roman Empire) in the fifth century to the Dark Ages in Europe and the subsequent decline in the level of intellectual development among the people.

We are aware that the Golden Age of Islam emerged seemingly out of nowhere, aiming to fill the intellectual void in Europe. It began in the seventh century, driven by the ambition to become the center for knowledge and learning for the world. This remarkable period is believed to have been fueled by advancements in paper-making technology and the exceptional translation efforts covering 'all secular/academic books' available during that time. This translation endeavor included works in Greek, Sanskrit, Syriac (which already had extensive translations of Greek medical books, among others), and more, spanning the eighth to the tenth century.

As a result, numerous academic and scientific discoveries unfolded during this period up to the thirteenth century, contributing to the Renaissance (beginning in the fourteenth century) and later the Enlightenment (commencing in the seventeenth century) in Europe. The decimal number system, featuring a positional system that includes zero, made its way to Europe in the thirteenth century from Islamic North Africa with Fibonacci. This event played a pivotal role in triggering a knowledge revolution in Europe. According to British writer and mathematics communicator Alex Bellos, the Renaissance was indeed sparked by the introduction of the (Indo-) Arabic number system, which includes zero.

The founding of the Holy Roman Empire (considered holy because it was sanctioned by the Pope and 'Roman' because it reunited parts of the last Western Roman Empire) in 800 CE on Christmas Day marked a new beginning in the role of education for humanity. The first emperor, Charlemagne, believed that the only way to keep his empire flourishing was through education. He transformed the character of the palace school and established similar ones throughout his empire. The focus shifted from religious education, primarily the translation and examination of holy texts, to a broader curriculum including grammar, rhetoric, music, arithmetic, geometry, and astronomy (known as liberal arts at that time). The formal inclusion of mathematics, specifically arithmetic and geometry, in schools was revolutionary. However, access to education remained elitist, limited to a fraction of society.

Incidentally, the creation of the explicitly Christian empire strengthened cathedral schools for clergy education in 'letters' to master reading and writing holy texts. Some of these schools evolved into universities, such as Oxford by the twelfth century and Cambridge by the thirteenth. Until the early nineteenth century, nearly three-fourths of Oxbridge graduates entered the Church. Among the rest, from the elite class - aristocracy or the wealthy, only a few pursued professions, and over 50% of Members of Parliament came from this group. Importantly, the emphasis on 'forming the minds of the elite' remains prevalent in many leading universities worldwide to this day. Of course, newer universities emerged from the Renaissance, Enlightenment, and the emerging middle class in the industrial society, focusing on practical arts and professions, as well as innovation and research.

The educational landscape in France leading up to the French Revolution was characterized by three significant features:

Centralization The educational system was centralized, with the church owning and operating most educational institutions. The focus was predominantly on religious education, emphasizing the translation and examination of holy texts. The origins of education can be traced back to the palace school, later extending to monasteries across the kingdom.

Ecclesiastical Control The church exercised authority over the scope and content of 'liberal' education. Teachers were mandated to obtain licenses from the church.

Elitist Access Education was accessible to only a small elite segment of society. The exclusive nature of students meant that only a fraction of the population had the privilege of attending schools.

These three characteristics continued to dominate education in France for centuries. The aspects of centralization and elitism in educational institutions persist to some extent even today.

Reorganising Mathematics Education
Locus of reboot is family

Mathematics has experienced remarkable expansion in its domains, coupled with increased foundational and procedural rigor over the past two centuries. For those interested, Cauchy's Real Analysis (1821) can be regarded as an epochal pinnacle in rigor. However, math education has not embraced such reforms; in fact, it has undergone changes that could be considered detrimental, such as the introduction of standardized tests and the dilution of mathematics curricula through the legitimization of 'quantitative literacy. The eminent French mathematician Henri Poincaré offers a critical perspective on the polarities in mathematics, emphasizing that studying the works of both great and lesser mathematicians reveals the presence of two opposite tendencies or, more precisely, two entirely different kinds of minds.

The two minds Poincaré was referring to are the logic-centric mathematical discipline and the intuition-powered force of reading mathematical relationships and objects in different situations. The logical mind adheres to rigorous applications of established mathematics, avoiding missteps in adventurous mathematising. On the other hand, the intuitive mind may seem to traverse seemingly unmathematical paths but ultimately withstands scrutiny through standard rigor, guided by conceptual integrity and reconceptualization prowess. Clearly, both kinds of minds have thrived among mathematicians and physicists alike.

Mathematics education must contexualise the content and nurture it as a sense to focus on building narratives and intuitive abilities. We need to reform mathematics education until we stop teaching it!

Humanising mathematics education – Mathematics as a right

Mathematical thinking is an equal gift at birth, we have made it hard to retain the same just past the unique childhood phase of human development.

Let us take a step back. How do six-month-old children learn to differentiate between music (pleasing to the ears) and noise (and cry in response)? How does one train a six-month-old baby to fall asleep to melodious music?

Do we teach drawing/colouring to eighteen-month-olds to express themselves? The drawings are always a medium to expansively communicate thoughts and imagination, unhindered by the demands of a natural language – words and structure.

Do we teach children how to walk? These, and more, are innate developmental milestones for every child. Similarly, learning the mother tongue is a natural ability. The simplest evidence is that all children automatically learn the language spoken by their mother, whichever it may be. This is equally true for a language called math; if mothers (and other involved adults around the children) converse with the children in it.

Indeed, learning to understand others and express oneself is very natural for children and is key to their survival (in fact, the offspring of all animals also have this ability). Being born human implies that it is also natural for us to effectively use math to express ourselves and be understood by others when conversing about quantities. However, the formal education of math often destroys the language, reducing it to a set of abstract algorithms and methods.

It is morally wrong to make an unnatural exception for math education. If we can ensure the introduction to and interaction with math in its true nature, then no child would struggle with math or fear it.

The real seat of learning for all languages is the home, the neighborhood, and the community. Unfortunately, we make an exception for math education by often ignoring family involvement.

This leads us to some of the most important questions about math education: Why is math not taught by parents or neighbors in the same way as drawing, music, playing/dancing/gestures, the mother tongue, and even a second language? Why do parents fail in enabling their children to learn math?

Simply put, math is another language—who may have had any difficulty in learning mother tongue? It should be no different when it comes to learning math. Math is a way to think about real-life and imaginative situations using a much smaller set of mathematical symbols and rules.

While all other languages like mother tongue, drawing, music, playing/dancing/gestures, etc., manifest and are nurtured to a great extent by the time children enter school, math waits for the school! The conversational competence reached by 4-year-olds in their mother tongue is variable; it depends on the diversity and richness of conversational language and experiences to which the child is exposed. On the other hand, the competence of pre-schoolers in mathematics is almost independent of their parents' educational and socioeconomic setting. For instance, all adults know estimation, comparison, counting, etc. However, most parents stay away from the conscious and concerted mathematics literacy of pre-schoolers. Mathematics education is entirely left on the shoulders of teachers and the mathematical competence of the authors of mathematics textbooks.

The very small minority of families that do introduce qualitative quantification, such as big, small, more, less, etc., leaves it at that. Such a gross introduction to the sense of quantity is actually an impediment in laying the foundations of mathematics as a very precise, extremely nuanced, and definitive language. For instance, there is little integration of pre-number experiences and vocabulary, familiarity with estimation through the Approximate Number System (ANS), and perceptual and conceptual subitising. Often, whatever the interaction with mathematics is, it is inappropriate in terms of invested time and diversity. And in all this, schools jump to counting using quantity numbers, not counting numbers, leaving children significantly misaligned to mathematical sense.

In case the severity of the math education deficit is not obvious, consider what would happen if schools, not parents, were to 'teach kids how to walk'. If it is still difficult to imagine the deficit in math education, just think of the relationship most of us had with math in our school days.

If you had been taught well, you would have fallen in love with math, and your life and career would have benefited from a happy association with math.

Yes, the way math is taught in schools is of inexcusably poor quality; the evidence is that being 'poor in math' is considered one sign of creative geniuses; also, it is the only socially acceptable 'failure' globally.

For the record, K–10 math curricula are no more than three academic years' worth of 'content'. Math is too rigidly hierarchical to be 'taught' for 12 years; formal math education is too slow, and this is hurting it.

The challenge of math education is growing with each generation; a bigger percentage of parents cannot support the math education of their children with each passing generation! This defies all logic and gravity!

Despite our most amazing achievements in every other domain of knowledge and practice, formal and informal math education remains an enigma.

It is unnatural for any child to be a 'slow learner' in math!

Humanity's cardinal mistake of *teaching* math

If the title is unbelievable and unsettling for most and kind of déjà vu for others, let it be upfront that mathematics is one domain of knowledge that cannot be taught at the school level. The act and fact of teaching mathematics in school years, especially the K-10 years, is humanity's undoing.

Curricularised, methodised, and abstracted math teaching has had the precisely opposite effect – push-back by children, every one of them in schools, and to the extent that 'math phobia' is peerlessly associated with some kind of pride, a badge of honour, a membership to an imaginary exclusive club, all over the world. Worse, most of us lose out on reading, play, art, music, social science, theatre, and above all, a happy childhood and parents chasing math because brilliance is equated with math in school academics.

We cannot sweat it out

Grade-wise curricula and syllabi, textbooks, assessments, gamification, 'math labs', lesson plans, and 'teachers' systematically destroy logical and mathematical thinking that is an innate human ability and an equal one among all infants. It can only be uncovered in a way that is wholly personalised for each child; foundational math cannot be subservient to standard, universal concept maps.

The more we have sweat in educating math, the higher the teacher centricity, the thicker the math textbooks, and standardised the assessments, the poorer the K-10 mathematical thinking. Math education, and then everything else, must be re-rooted.

Intellectual incapacity of even dreaming wholesale change

Math quite characterises the blind spots in human evolution – blame the children, childhood is superfluous, and inequity is natural.

UK PM's clarion call and commitment to ensure math education for all till 18 years of age is making waves in the power corridors across the world. He emphasized the need for a profound shift in mindset in education today, urging a reimagining of our approach to numeracy. The Prime Minister stated that ensuring every child receives the highest possible standard of education was the single most important reason for his entry into politics.

However, the PM's office clarified that the government does not foresee making Math A-Level compulsory for all 16-year-olds. Unfortunately, and still inexplicably, improvements in math education can only exhibit a limited degree of boldness and ambition. Math education remains somewhat untouchable, still far beyond control across nations and even in the best universities, afflicted with paralysis of will and intellectual incapacity to think beyond improvisations.

Nearly 200 years of shame

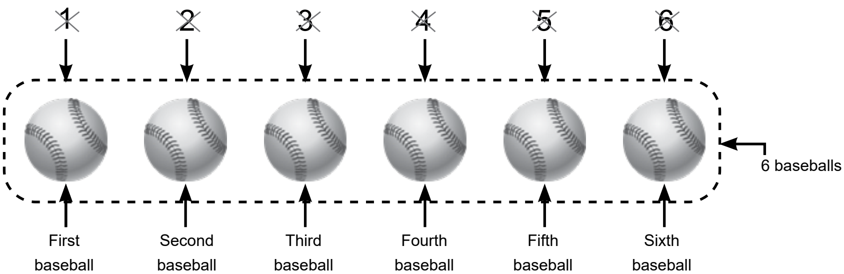
The most certain, vivid, and personal experience and fact around the current school system – K-12 – is the secular trend of decline in math achievement of pupils across the globe. This is despite continuous interventions and innovations in math education processes and resources; there is something fundamentally wrong with math education in schools.

With all modesty as a school educator and mathematician, but fired by a definite sense of angst and loss, the writing has been on the wall in all the years since the current form of institutionalisation of education of children around the mid-nineteenth century. Here is just a sample of the hundreds of wrong, confusing, soulless, 'groundless', and even illogical contents of teaching math in schools:

School math has never really corrected or modified itself to explain how and why '8 divided by $\frac{1}{2}$' is 16 because it is not; it is 16 $\frac{1}{2}$s (16 halves), just as '8 divided by 4' is 2 4s (2 fours)! 16 is the wrong meaning of '8 divided by $\frac{1}{2}$'; it is also anti-science in the sense that 8 things can never be equal to 16 things. And the explanation of 'taking the 2 up and multiplying with 8' is utter nonsense.

1, 2, 3, 4, 5 … are used for counting, instead of First (I), Second (II), Third (III), etc.

Pictorially,



Counting up to six

Interestingly, counting (ordinal) numbers come from natural languages, not math (quantity or cardinal numbers originate in math). Introducing and using ordinal numbers for counting would significantly simplify understanding of quantity and operations for children!

School math contains poorly-delineated, unexplained, joylessly presented, and soulless concepts and methods. For instance, factors of a number are taught to be obtained so that when multiplied, they give the number as their product; 2, 5, and 7 are the factors of 70, as $70 = 2 \times 5 \times 7$. There is no explanation for why the original number can be obtained by multiplying the factors and why addition, subtraction and division cannot be used to find 'parts' of the number that are the factors. Incidentally, a beautiful explanation underlies why the other three operations cannot be used as factors, and the explanation brings forth the simplicity and lucidity of math.

It is groundless, baseless, to teach 'borrow' and 'carry' as properties of operations (subtraction and addition, respectively); the two are natural 'implications' of the positional number system (the decimal number system).

Another example of baseless (and wrong) teaching of mathematics is the commonality in writing, speaking, and most typical understanding of $\frac{3}{5}$ as a fraction and $\frac{3}{5}$ as a division – both are considered the same (0.6, for example). But $\frac{3}{5}$ as a fraction is one number and quantity, and 3 and 5 are not distinct but together, and $\frac{3}{5}$ as a division has 3 and 5 as separate numbers and quantity and represents two quantities and numbers – a quotient and a remainder.

School math, in Grade III or IV, teaches

II (Second) + IV (Fourth) = VI (Sixth)

This is patently illogical. The point is best understood through an everyday example – If I own the second-floor and fourth-floor houses of a multi-storey building, I own two houses, not the sixth-floor or six houses!

It is inexcusable that except for some enthusiastic teachers and some national curricula, math education remains riddled with a few hundred holes of the abovementioned kind. Of course, even these efforts are eventually impactless on students because they are nowhere curricula-wide.

Mathematicians missing the faux-math at schools

Mathematicians are a class by themselves as individuals and professionals. They are personally known to be private, geeky, and even reclusive. Professionally, they have the unique privilege of choosing the challenges, setting their goals, and operating in intensely intellectual, specialised, and committed conditions. The work of mathematicians is almost always an input to other mathematicians or leading-edge researchers.

Apparently, and not unexpectedly, the irrationality of disastrously poor outcomes of math education in K-12 has not rattled mathematicians. It is crucial to deliberate that if there is nothing wrong with the children or mathematics, then math education is flawed.

How may any (normal, average) human child struggle in math, a wholly logic-based knowledge domain? Human children cannot be struggling in math.

No mathematician seems to have publicly and vigorously argued against the fauxness of math in school, as illustrated in just the previous section. Mathematicians and the fraction of students who do crack the code of the faux-math in school (such as 8 is 16 as 2 of $\frac{1}{2}$ in the denominator goes up for multiplication with 8) bask in the glory of being bright, and more intelligent and leave things there for all others.

Almost all of us have missed the 'mathematical sense' – mathematical thinking, the language that is math, and the everyday realness of mathematics. The mathematics known to mathematicians – the abstracted one – is handed down to the KG students; the state of math education is no surprise. Of course, abstraction is at the heart of math, being the language of patterns and the orderly universe, but K-12 math better be just real, natural.

Mathematics (education) is a massacre of rationality

It was, and remains, a parody of Charles Dickens's Hard Times, of children as 'little pitchers' to be filled so full of mathematics. Ironically, that is how to introduce and cultivate any kind of habit, skill, or knowledge new to the target audience.

Thus, if wholly decontextualised and abstracted mathematics and mathematics education were right, math education should have worked over centuries.

However, if mathematical sense and thinking are evidently in us, we must end the current design now. Indeed, it is strange that using computing devices is still not allowed in mathematics classrooms, as well as assessments and examinations.

Mathematics reboots educational opportunities

Mathematics as a language offers the most stunning developmental trajectory – it puts all 8 billion of us today on the same pedestal in a language that is the language of AI – the fast-emerging people's 'meta-language'! However, (real) mathematics is alien to us all; it is a way of thinking that was killed for us all in pre-school itself.

Children excelling in mathematics is only a function of intent, involvement, and effort at the family and community level in using mathematics as a language. It has little to do with educational 'levels' of parents and communities. This rebooting of equal scholastic opportunity for all children is an opportunity to put us all on a sustainable growth path. Of course, the privileged amongst us are not disadvantaged as they can learn the language of mathematics faster than most others.

We cannot imagine any other opportunity for humanity in this century to deterministically steer us in a new direction that gives us hope and assurance of survival and growth.

Future of math education in school

Math education should move beyond classrooms and must become the responsibility of families. Schools may be used for providing a community of co-learners, tutoring support to those who demand it, and to handhold parents in a similar way.

As mentioned, mathematics as a language is as alien to schools and teachers as to parents. Parents would have to learn mathematics anyway, even if schools were to lead the change to mathematics education. The best course would be for parents to learn and take the lead in changing the richness of mathematical discussions at home and in the community.

Who cares?

'What is math?' remains unanswered in 14 years of school education. Worse, a good majority of parents, students, teachers, school leaders, and policymakers do not even want the answer; math is 'not for them'. Ironically, even the majority in the minority who care for math education pursue math education for instrumental values – a better career! And there is more – for the exceptional few, for whom math is pure joy, math is one of their love affairs for life; expectedly, 'what is math' is not really relevant to them.

Parents should care

Yes, we need to make a new beginning in math education. We must 'discover' and agree on 'what is math?' and widely disseminate it across the globe. It is interesting to see other evidence of math as the only universal language – 'What is math' is a global blind spot; no nation has got it right! Refounding, rethinking, and redesigning school math education is a humanity-wide challenge. Naturally, my elaboration of 'What is math?' must be seen in this context as a contribution in this direction. My story of 'What is math?' is also shaped by my target audience – parents. Parents at large would anyway be left out of formal (re)education on 'what is math?', as educators and policymakers are the only targets of such efforts. I consider my focus on parents as the more critical contribution to math education.

Children will also be parents

After at least five generations of adults educated in school (in about 175 years of the modern, mass form of school education prevalent today), most of us cannot confidently teach math to our children in the primary grade. It hurts to live with the feeling that nothing better can be expected of the children in schools today. It is less likely that there may be adults who would joyously teach math to the children in primary grades!

Let us learn the 'real math', a thinking process and resource that expands and sharpens with every living moment!

Philosophy of Mathematics Education
The weakest link

A philosophical lens on any domain of knowledge enhances its capacities to organise ideas and issues, thereby better presenting concepts, definitions, arguments, and problems.

A clear larger issue in K-12 mathematics education is the balance between contextualisation (mathematics as a language - creative and intuitive play) and de-contextualisation (abstracted and procedural rigour). And the balance is not about any absolute choices; it is a dynamic choice; for instance, grade, topic, practice, and remediation are just some of the determinants of the extent of balance. Relevantly, the dominant position of contextualisation of mathematics is foundational in many ways. The US civil rights activist Malcolm X hits the nail on the head about the facelessness of the current, abstracted mathematics education when he expressed regret, saying that the subject he most disliked was mathematics. Upon reflection, he believed the reason was that mathematics allows no room for argument. If a mistake was made, that was all there was to it.

However, it would be unfair and wrong not to emphasise the role and importance of rigour in mathematical research and applications – for instance, precise and universal definition of concepts, sound methodological presentation, and established procedures. The rigorous reinterpretation of the calculus undertaken by Augustin-Louis Cauchy in the early nineteenth century was indeed a major intellectual revolution in calculus, larger mathematics, and science.

It started the ever-growing place of calculus and mathematics in our lives and changed the development of science and technology forever. In fact, Leibniz, the famous 'co-founder of calculus', is known to have talked about how his 'algebra based calculus' had "outstripped the methods of traditional mathematics." Cauchy articulated that he endeavoured to provide methods with the rigour essential in geometry, eliminating the need to resort to rationales based on the generality of algebra. To be simple, this was also Cauchy's tribute to Euclid for the rigour he had brought to geometry and mathematics over 2,000 years earlier, the first of such attempts that has lasted to date.

However, we have little reason to be dragged into this debate. In respect of this book, the interest is limited to mathematics education in K-12.

The context

Philosophy of mathematics is a flourishing domain of knowledge, though what it does to mathematicians' work is another matter. The philosophy of mathematics has remotely impacted the philosophical direction and content of mathematics education.

It is a contradiction to even attempt a well-grounded and evolving philosophical substrate for something that philosophy finds hard to circumscribe ('math is hard to be comprehensively penned for its present and futuristic focus and impact'). And when the majority views mathematics to be predominantly about 'abstract structures'.

No wonder the default goal and means of mathematics education is abstractness, starting well into the pre-school and pre-primary years. The ill-foundation of mathematics as black or white, correct or wrong, right method or the wrong method, uncritical and uni-solution-focused thinking, is laid within weeks of entering the portals of the formal education system. For instance, approximation as an authentic and necessary mathematical experience and structure for quantification is killed with the quick introduction of definitive counting for quantification.

Indeed, there is a good body of educationally sound and biologically gifted knowledge of approximate number systems and subitising; it is proven a priori knowledge, for we share it with animals. Many school systems, including those in India, almost miss this bus entirely, and all others need to do justice to adequately hone the sense of estimation/approximation.

Next, counting misses a critical beat and gallops into an unfathomable ditch. All of mathematics is about quantification, the most superior input to decision-making (for the most part), and counting is all that is. All mathematics, starting from the arithmetical operations, is about making counting faster. For instance, addition is faster in quantifying a given set of things than counting the same, multiplication is a faster quantification process than addition (when adding all the same quantities), the exponent is even quicker, and using logarithm is the fastest we have (it is the same order of relative time for computers too).

School systems reduce counting to 1, 2, 3 …, a gross misrepresentation of counting; 1, 2, 3 ... are not counting (ordinal) numbers but quantity (cardinal) numbers. Counting numbers are First, Second, Third, …; elsewhere, we have detailed how unfortunate is this global blind spot, yet there is more – the foundational idea of 'unit of counting/quantity' is lost in this process and plays havoc with children's quantitative visualisation. More relevantly, this state of affairs is in good part due to the lack of simple symbols for ordinal numbers and children that young cannot write 'First', 'Second', etc.

An essential cause of this wrong is also the fact that counting numbers is rooted in the natural languages; the quantity numbers are the creation of mathematics; unsurprisingly, mathematicians and mathematics educators fail to borrow the counting number names to mathematics. Suppose some of you are thinking about using Roman numbers, an ordinal number system, to learn to count right. In that case, you are in the right direction – we need to make a simpler version of Roman numbers (discounting V, X, … – kind of the more ascriptive symbols).

The stance needed, philosophically

Reducing the larger philosophical dimensions of maximising learning outcomes into two – reasoned conceptual clarity with extensively contextualised narrative or rigorously structured logical relationships to deconstruct any mathematical context. This must be read in the context of mass-scale, extensive quantitative literacy versus scaling up school mathematics curricula to support better higher education.

The story of Calculus is very illustrative here. It is no less profound than the stories of the birth of Calculus at the hands of Newton (and Leibnitz) in the later XVII century. It is about the opposing philosophical account of how to make the understanding of calculus easier and more vibrant.

Nearly a century and a half later, in 1821, Augustin-Louis Cauchy's book, the *Cours d'analyse* (considered one of the most influential mathematics books ever written), offered the most expansive definition of limit. Just two decades later, Augustus De Morgan authored *Differential and Integral Calculus* and focused on conceptual clarity rather than the best definitions (the book famously had a 29-page introduction).

For this reason, we have also appended a uniquely conceptual yet rigorous introduction to calculus, differentiation, and integration built only on the concept of limit and continuity but not on the arithmetical methods that have been crying for reform of calculus education for the past 200 years (since the mathematician Cauchy's re-founding calculus on limit).

Annexure

Mathematisation Case Studies

Trigonometry and Calculus

Seeing is believing. Experiencing is truth. Let us experience 'mathematised Trigonometry and Calculus'.

Pertinently, among the more unique and important features of mathematics as a language and a domain of knowledge is that it has come to be so abstracted, regourised, and procedured for the sake of widest applicability that it is too expansive even for mathematicians. We, the authors, are not mathematicians, and despite that handicap we assert that of all kinds of researchers mathematicians follow the most specific interests (of course, that also implies that their contributions are most impactful for humanity.) Education of mathematics needs to be revolutionized.

The abstractness of trigonometry is widely acknowledged. John G Kemeny, a remarkable mathematician and computer scientist, questioned the relevance, stating that a considerable portion of his high school trigonometry course was dedicated to the solution of oblique triangles. However, he expressed that throughout his highly varied career, he never found an excuse to use these techniques and questioned the necessity for all high school students to devote several weeks to the subject.

On the other hand, calculus education misses out the beauty and the beast that it is. In the words of the mathematician Steven Strogatz, calculus insists on a world without accidents, where one thing leads logically to another. With the initial conditions and the law of motion, calculus allows us to predict the future or, better yet, reconstruct the past.

Mathematisation of thinking

Mathematisation of thinking is building natural-language-like competence in expressing real or imagined relationships of quantities. It would help to know that this mathematisation is best raised on high proficiency in the chosen language of academics. For, language mediates thinking, and a brain that is already accomplished in using abstract objects and constructs – words, syntax, grammar, semantics, morphology – is far better equipped to master another language. Mathematisation of thinking is to harness mathematics as a language to comprehensively and uniquely visualise and express situations involving quantities.

Interestingly, the benefits of mathematics as a language are well appreciated. But it quite ends there; it is not practised, not even among mathematicians. The reason for this divorced state of possibility and practice is very illuminating. Natural languages are so-called because learning them is simply by participation; just being all ears requires registering literal correspondence between the words and the objects/feelings they represent. The formal constructs of our first natural language – the mother tongue – are for literary writing capabilities, not for accomplished communicative writing or reading literature in that language.

However, a layer of formal learning is required for 'non-natural languages', or acquired languages, such as mathematics, art, music, dance, 'theatrics', games and sports. All these languages are somewhat innate, a kind of sense/knowledge, and thus, too personal, and need to be framed into a common framework for communication with others. For instance, music is so highly structured/framed that it is almost universal; good music is pure science (and mathematics). Music is all sounds that are pleasant to our brain; all else is noise; that is why music is quite universal.

Mathematics must also be structured to be effective as a means of communication. We already know how mathematics has the simplest demands and complexities as a language. There is little by way of convention in mathematics (for example, the way we write numbers draws Cartesian planes), and even the list of standard notations is not much. What is the twist in the story of mathematics that makes it a 100% precise language (every physical reality has only one mathematical expression) and 100% universal? The simple answer is what we call concepts and the rigid network and hierarchy of concepts. Learning and mastering these concepts needs formal education and their application in the routine.

Thus, the mathematisation of thinking boils down to intensively exploring the conceptual foundation of the various dimensions of mathematics. This implies a significantly toned-down role and place of 'rigorous, calibrated mathematics' in mathematics as a language. To be convincing, we have chosen two dimensions of mathematics – (secondary level) Trigonometry and (senior secondary level) Calculus – as case studies of conceptual exploration of mathematics.

We hope that a good read of the two case studies would see you falling in love with triangles and Calculus and imbue you with newfound lenses to critically and creatively quantify disparate everyday and professional contexts, thereby setting off a new relationship with mathematics and the world because the two are also the most challenging of mathematics in K-12. The contrast with the extreme abstractness of school mathematics should be apparent, and the place of 'process and proof-driven' scholastic mathematics may be respectfully questioned and revisited. We expect that the real nature of mathematics will be revealed and mathematics education will become the fountainhead of AI-age thinking humans.

An introduction to 'Mathematised Trigonometry'

Trigonometry is placed on the cusp of secondary and higher secondary education and is literally the high point of secondary geometry, algebra, sets, and functions. It is also the best closure to the enigma that is triangles, in terms of the lion's share of geometry curricula up to secondary years. Yet, trigonometry education cannot be more recklessly designed and delivered.

Trigonometry is best introduced and internalised as a function, a special relationship between the angles and sides of triangles. Understanding trigonometry as a set of functions that dramatically simplifies visualisation and verbalisation of the three 'primary trigonometric functions' – sin(e), cos(ine), and tan(gent) – and their multiplicative inverses – sec(ant), cosec(ant), and cot(angent).

Even better, senior secondary's nemesis – the inverse trigonometric function(s), especially when coupled with calculus – is a delight to be introduced as a function.

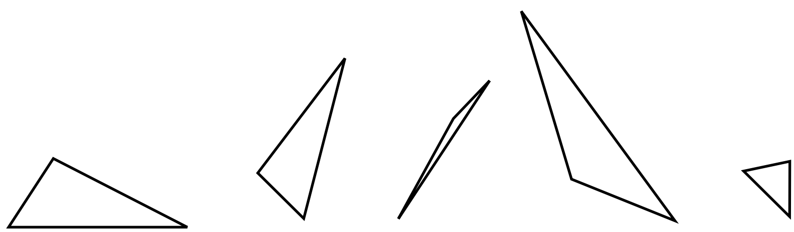
To be fair, relations and functions are often out of the secondary syllabi, and using these to explore trigonometry would not be possible without curricular reorganisation. But that reorganisation is anyway an imperative for another reason too – knowledge of sets is integral to counting, and quantification. Sets must be introduced in the pre-school years and better explored in the primary school years.

By the middle school years, interactions among sets could easily be studied at the basic level. Relations and functions are the operations through which sets interact. Relations and functions are the gateway to many mathematical foundations.

Besides, functions play a pivotal role in calculus. The latter would not be possible without the use of functions.

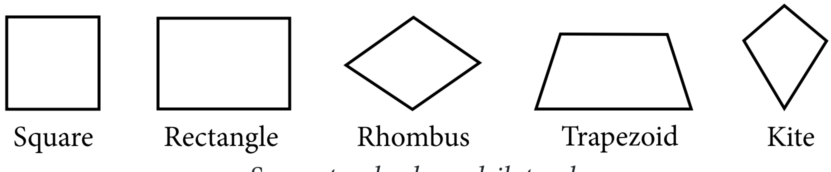
Mathematised Trigonometry

Triangles, the simplest polygon we come across in everyday life and geometry have a wide variety of shapes and properties depending on the measure of the angles and the length of the sides. It would be hard to think of a 'standard' or more common kind of triangle, that is, a more common shape of triangles.

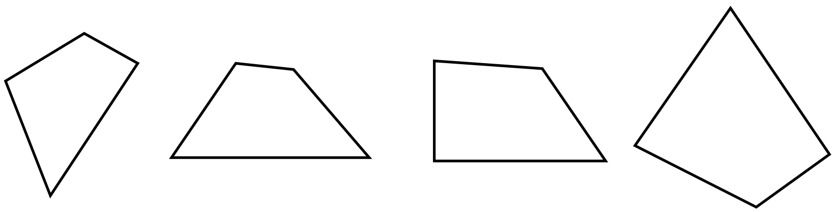


Triangles of different shapes

On the other hand, when it comes to the other polygons (with four or more sides), knowledge of measures of just the sides won't identify a unique polygon. For example, for a quadrilateral, we need to know the four sides and the diagonals too to make a unique quadrilateral. That is why, we have special quadrilaterals – parallelogram, kite, rhombus, etc. – all limited by special conditions, such as all angles 90° and all sides same, for squares.

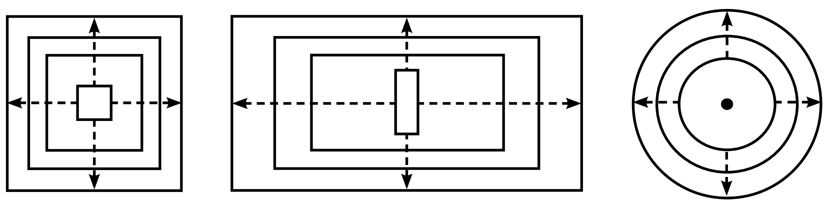


Some standard quadrilaterals



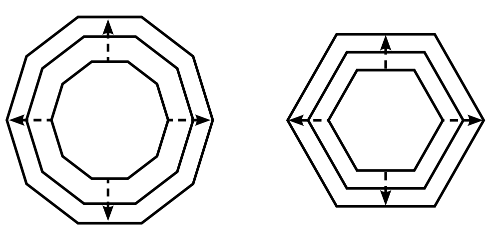
Uncommon quadrilaterals

The standard quadrilateral does not exhibit many variations, for example, squares and rectangles vary only with respect to the length of the sides. Thus, their shape changes to become bigger or smaller. Similarly, depending upon the length of the radius, a circle changes to become bigger or smaller only.



Size of standard quadrilaterals and circles differ in length, not in shape

The same is true for other kinds of polygons, such as pentagons, octagons, decagons, etc. In polygons, the standard form has sides with equal length. Such polygons are called regular polygons. There is no difference in the shape of the different versions of these standard (regular in geometry) polygons with the same number of sides. All pentagons, hexagons, and decagons, for example, are just bigger or smaller sizes of the same shape.



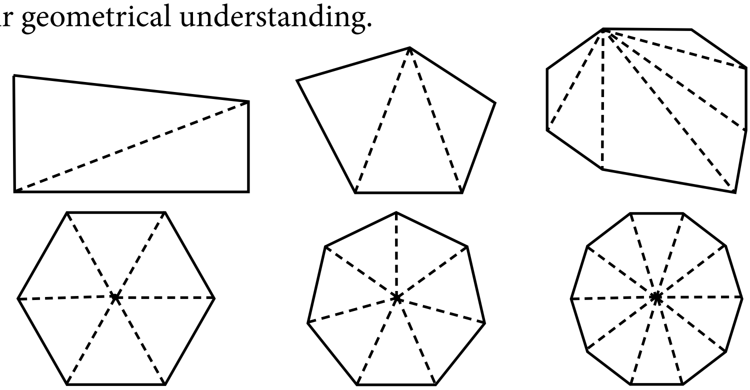
Size of standard polygons differ in length, not in shape

Triangles are unique polygons. They have innumerable variations.

Welcome to the diversity and the beauty that is triangle

The fact that triangles are polygons with the least number of sides and come in very different shapes is a boon in geometrical analysis. All kinds of polygons (four sides or more – whether regular or irregular) are geometrically studied by visualising and decomposing them into multiple interconnected triangles– their fundamental building blocks.

This also reinforces why we must study triangles in all their diversity, and they better not be reduced to any common forms for their geometrical understanding.



Polygons are made up of multiple triangles

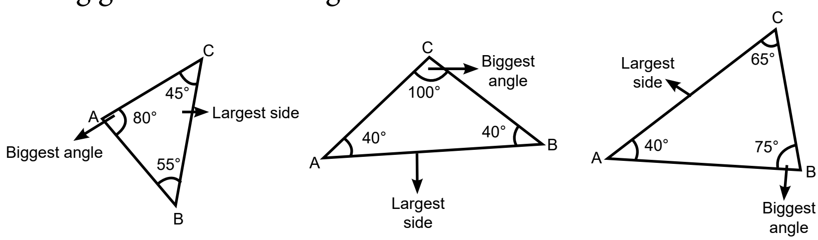
The root of trigonometry

Expectedly, there is a branch of mathematics that is dedicated to the study of triangles. It focuses on how the measure of angles and length of sides help compose infinitely unique triangles. To that end, it studies the relationships between the length of sides and the measure of angles of triangles.

The saving grace

Thankfully, despite the apparent diversity in the shapes of triangles, their properties reveal a remarkable simplicity when it comes to discovering patterns in the relationships between angle measures and side lengths.

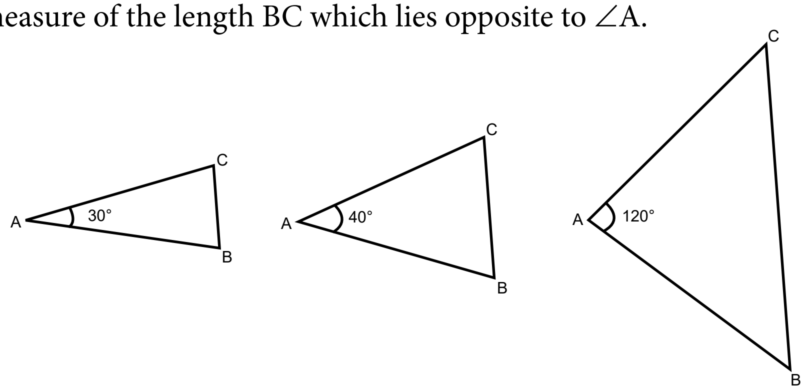
In triangles, a significant and indisputable relationship exists between the measures of angles and the lengths of the sides opposite to them. This fundamental connection acts as a saviour when tackling geometric challenges.



Angle opposite to the largest side in a triangle is always biggest and vice-versa

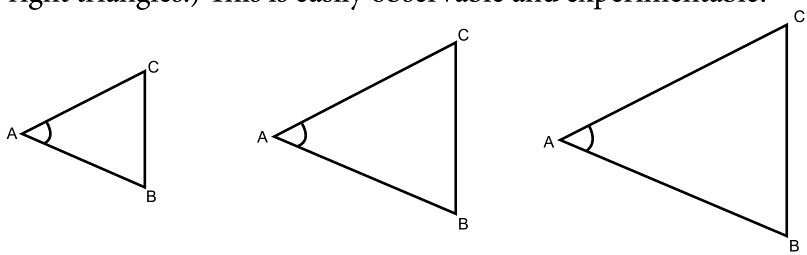
However, this relationship turns out to be specific – a given angle measure does not have a fixed length of side; it is limited to the fact that any increase in an angle measure will lead to an increase in the length of the opposite side (whatever the length is).

As the measure of ∠A increases, so does the increase in the measure of the length BC which lies opposite to ∠A.



Increase in measure of angle means length of its opposite side increases

And, even that does not always hold - for the same angle, the opposite side could increase or decrease in length! Does it violate the nature of the aforementioned relationship? Yes and No. Yes because there is no direct sides and angles relationship. No because we will soon come to observe and easily experience that the finer, universal relationship of angles and sides is in the form of the relationship of angles and ratio of sides. This is broadly intuitive in the following pictures that show how the other two sides of a triangle also change if the opposite side to a fixed angle is changed. Fortuitously, in right triangles there is a consistent relationship of angles and the ratio of length of sides (in all right triangles.) This is easily observable and experimentable!



Same measure angle with different lengths of the opposite side has different lengths of the other sides

Indeed, in triangles the three sides and the three angles need to be studied extensively to understand its different facets.

Welcome to trigonometry, the domain of mathematics that helps us to measure all the six measurable dimensions of triangles – the three angles and the three sides.

A note on learning about Trigonometry

The documented roots of trigonometry can be traced back nearly 2500 years ago, and it likely has an even longer history that dates back further. Trigonometry emerged from astronomy in ancient civilizations as a practical tool for studying celestial objects.

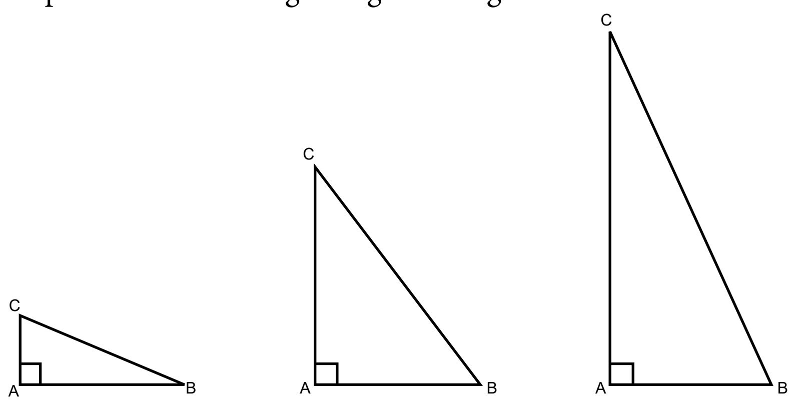
The geometry of celestial objects is 3-dimensional, not planar (2-dimensional, or what is called Euclidean geometry), and thus, the corresponding trigonometry is spherically oriented, not planar trigonometry.

At the best of K-12 geometry, spherical geometry (such as the shortest distance between two points on the surface of the earth such as the 'straight flight path' of migratory birds is not straight, it is an arc on the spherical surface of the earth.) is not part of curricula. And it need not be, planar geometry itself is a huge part of our lives, science and engineering too.

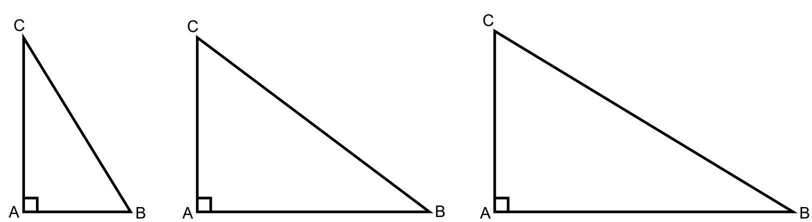
To the point, the foundational concepts in K-12 Trigonometry are made unduly complex by using spherical geometry for just studying planar trigonometry. For example, trigonometry was founded with 'trigonometric functions' in terms of arcs/chords of circles, but it does not mean K-12 education has to use the same foundations. As we will experience soon, functions (which are central to mathematics in all that is the real world) are a much better way to understand trigonometry.

Thus, we will explore the foundations of trigonometry from an easily visualisable and logically threaded narrative using functions.

The K-12 trigonometry also makes one more simplification – we study trigonometry for right-angled triangles only; this makes learning about sides and angles easier because the possible variations in the shape of triangles are dramatically simplified (still infinite in numbers). The following pictures show how there are only two kinds of shape variations in a right-angled triangle.



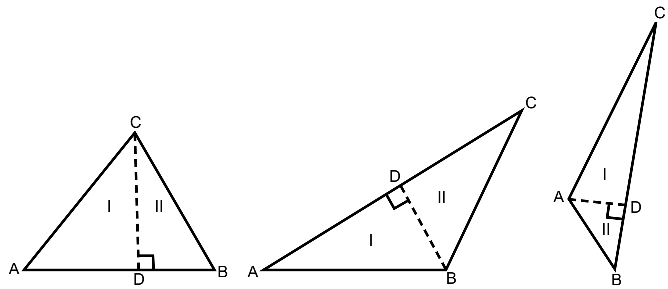
Hypotenuse increases with the increase in height of a right triangle with same base length



Hypotenuse increases with the increase in base length of a right triangle with same height

Remarkably, this simplification is similar to how regular polygons and circles are simpler shapes.

And the best news – there is no compromise in applying trigonometry as all triangles can be seen as composed of two right-angled triangles.

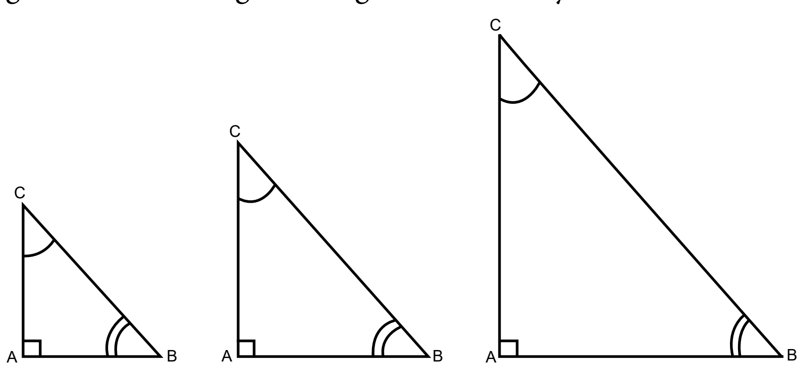


Every triangle consists of two right-triangles

Trigonometry

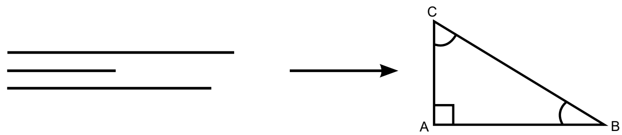
There is a very interesting fact about all the angles and all the sides of a triangle.

We cannot find the sides of the triangles even if we know all the angles – the same-angled triangles can have any-sized sides.



Triangles with equal angles but varying sizes

But the other side of the question is doable – we can find the angles of a triangle whose three sides are known. For example, if three lines of any length are given, then a definite right-triangle is possible with the three lines (as long as inequality theorem holds).

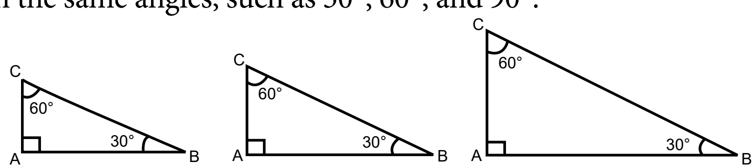


Right-angled triangle can be created using three lines of any length

We can now get down to expressing the (precise) mathematical relationship of angles and ratio of sides of right-triangles. Broadly, it's three scenarios: - finding sides when all the angles are known, finding angles when all sides are known, and finding unknown sides and angles when a mix of sides and angles are known.

First scenario

Knowing all three angles doesn't give a specific triangle—just a shape. Infinitely many triangles of different sizes share those angles; the only feature of the sides that holds across similar triangles is that the ratios of corresponding sides remain the same. However, in all right triangles with the same set of angles, we notice another striking feature: the ratios of any two sides in all such triangles also stay the same — forming the foundation of trigonometry. To be sure, with three known angles, the closest we can come to knowing about sides is their ratio, not actual lengths. Closely observe the following triangles with the same angles, such as 30˚, 60˚, and 90˚.

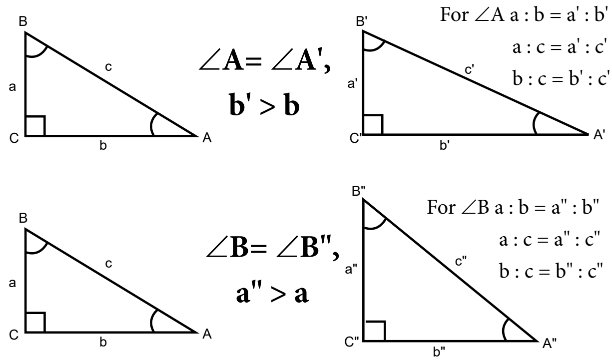


Triangles of varying sizes with angles 30˚, 60˚, and 90˚

Visibly, the one truth about the sides is that the ratio of any two sides of one triangle is equal to the ratio of the corresponding sides of other triangles. For example, $\frac{AC}{AB}$ in all three triangles would be similar.

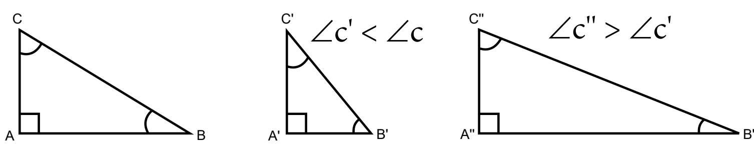
The same is true for $\frac{BC}{AB}$ and $\frac{BC}{AC}$.

And, the following pictures show the 3 ratios of sides remain the same if only the size of the triangle is changed.



Most importantly, these pictures show that the 3 ratios of sides of right-triangle did NOT change as the sides changed, as long as the angles remained the same!

Thus, a given angle always gave the same set of 3 ratios of the sides whatever be the size of the right-triangle (i.e., whatever be the size of the sides.) This kind of definite relationships are called functions!
To be sure, a change of angle will change the 3 ratios:

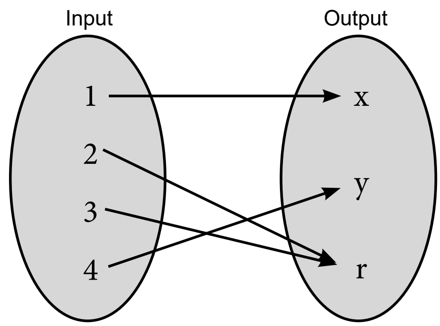


$$\frac{AC}{AB} \neq \frac{A'C'}{A'B'} \qquad \frac{AC}{AB} \neq \frac{A'C'}{A'B'} \neq \frac{A''C''}{A''B''}$$

Mathematics is a beautiful and powerful knowledge also because it invented functions. ‘FUNCTIONS’ take some input quantities and ‘process’ them to get an output quantity. They convert one kind of quantity into another. We would use functions which will convert angles into ratio of sides.

Mostly, functions are explicit and quantitative relationships between two or more quantities. A function defines how one quantity (the dependent variable) depends on one or more other quantities (the independent variables).

One of the distinctive features of a relationship that is a function is that for every set of input(s), there is only one and only one corresponding output quantity, and it will always be the same for those inputs. In other words, a function assigns a definite output value to each (set of) input value(s).



A function is a relation that assigns to each input exactly one output

The functions that relate the measures of triangles (which are only of two kinds – angles and sides) are called trigonometric functions; recall, trigonometry is the study of the relationships between the measure of angles and length of sides of triangles. Importantly, the trigonometric functions are applicable to all kinds of triangles, not just to the right triangles because all kinds of triangles bear a direct relationship between their angles and sides.

We are focusing only on the trigonometric functions applied to the angles of right triangles because it is simpler to study right triangles. Also, all non-right triangles could be seen and studied as two right-angled triangles. That's why, the school syllabus focuses on the trigonometry and trigonometric functions of right triangles.

Trigonometric functions take angles as inputs and produce the ratio of relevant sides as output. We have already explored how angles of triangles bear a direct relationship with the ratio of sides, rather than just the length of one side.

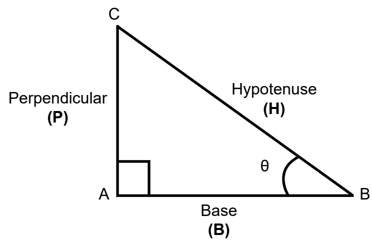
The term 'trigonometric functions' is indeed a more encompassing and intuitive name for these mathematical functions, especially when compared to referring to them as 'trigonometric ratios' (the more common name in school textbooks).

However, there is a natural query arises – what if the ratio of sides is known and the relevant angle needs to be determined?

We can use the inverse trigonometric functions to find the angle when we know the ratio of the sides.

Inverse trigonometric functions take the ratio of the length of sides of the triangle as their input and produce the relevant measure of the angle of the triangle.

Let us start with finding out how many ratios of sides exist in a right-angle triangle. Consider the following right triangle.



A right-angled triangle ABC

There are six possible ratios of sides in the triangle –

$$\frac{P}{H}, \frac{B}{H}, \frac{P}{B}, \frac{H}{P}, \frac{H}{B}, \frac{B}{P}$$

We need six different functions, which when applied to the angles of the triangle give us six ratios as their outcomes.

It is easy to appreciate that in all right triangles, one angle is always 90°, and any two right triangles are different only in terms of the other two angles (θ, ϕ in the given figure). Thus, all the ratios of the length of sides of a triangle are linked to a distinct function with respect to an angle (θ or ϕ).

Here is a table of six trigonometric functions of angle θ.

Assumed Function Name	**Ratio of sides**	**Trigonometric Function Name**	**Trigonometric Function (of angle θ)**
First Function of angle θ	$\dfrac{\text{Side opposite to } \angle\theta}{\text{Hypotenuse of the triangle}}$	**Sine** Function of angle θ	$\sin\theta = \dfrac{P}{H}$
Second Function of angle θ	$\dfrac{\text{Side adjacent to } \angle\theta}{\text{Hypotenuse of the triangle}}$	**Cosine** Function of angle θ	$\cos\theta = \dfrac{B}{H}$

Third Function of angle θ	$\frac{\text{Side opposite to } \angle\theta}{\text{Side adjacent to } \angle\theta}$	**Tangent** Function of angle θ	$\tan\theta = \frac{P}{B}$
Fourth Function of angle θ	$\frac{\text{Hypotenuse of the triangle}}{\text{Side opposite to } \angle\theta}$	**Cosecant** Function of angle θ	$\text{cosec}\,\theta = \frac{H}{P}$
Fifth Function of angle θ	$\frac{\text{Hypotenuse of the triangle}}{\text{Side adjacent to } \angle\theta}$	**Secant** Function of angle θ	$\sec\theta = \frac{H}{B}$
Sixth Function of angle θ	$\frac{\text{Side adjacent to } \angle\theta}{\text{Side opposite to } \angle\theta}$	**Cotangent** Function of angle θ	$\cot\theta = \frac{B}{P}$

Similarly, we can link all the six ratios of length of the sides to the six trigonometric functions of angle ϕ.

The multiplicative inverse functions of sine, cosine and tangent

The six trigonometric functions of angle (θ or ϕ) have their inter-relationship in terms of the ratio of the sides.

The cosecant function of angle θ is the reciprocal of the sine function of angle θ. Similarly, the secant and cotangent functions of angle θ are reciprocal of the cosine and tangent functions of angle θ respectively.

Thus, the cosecant, secant, and cotangent are not the inverse functions of the sine, cosine, and tangent. They are the multiplicative inverse functions.

Multiplicative Inverse Functions	**In terms of Sides**	**Result**
$\text{cosec}\,\theta = \frac{1}{\sin\theta}$	$\frac{H}{P} = \frac{1}{\frac{P}{H}}$	$\text{cosec}\,\theta \times \sin\theta = 1$
$\sec\theta = \frac{1}{\cos\theta}$	$\frac{H}{B} = \frac{1}{\frac{B}{H}}$	$\sec\theta \times \cos\theta = 1$
$\cot\theta = \frac{1}{\tan\theta}$	$\frac{B}{P} = \frac{1}{\frac{P}{B}}$	$\cot\theta \times \tan\theta = 1$

Second scenario

All angles can be found if the lengths of all sides are known. But how can we do it mathematically without actually measuring the angles? We can use the inverse trigonometric functions to find the measure of angles when we know the ratio of the length of sides.

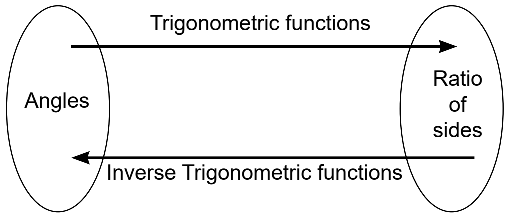
Inverse trigonometric functions take the ratio of the length of sides of the triangle as their input and produce the relevant measure of the angle of the triangle.

The inverse of the sine, cosine, and tangent functions is written as $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

Trigonometric Functions of an Angle is Ratio of Sides	**Inverse Trigonometric Functions of Ratio of Sides is an Angle**
$\sin\theta = \frac{P}{H}$	$\sin^{-1}\left(\frac{P}{H}\right) = \theta$
$\cos\theta = \frac{B}{H}$	$\cos^{-1}\left(\frac{B}{H}\right) = \theta$
$\tan\theta = \frac{P}{B}$	$\tan^{-1}\left(\frac{P}{B}\right) = \theta$
$\operatorname{cosec}\theta = \frac{H}{P}$	$\operatorname{cosec}^{-1}\left(\frac{H}{P}\right) = \theta$
$\sec\theta = \frac{H}{B}$	$\sec^{-1}\left(\frac{H}{B}\right) = \theta$
$\boldsymbol{\cot\theta =} \frac{B}{P}$	$\cot^{-1}\left(\frac{B}{P}\right) = \theta$

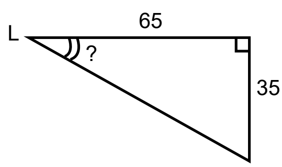
Trigonometry itself is quite a big deal, and inverse trigonometric functions simply scare us all, even in Grades XI-XII.

But, as we see above, inverse trigonometric functions are just another way of expressing the ratio of sides! Of course, the inverse also simply asserts that just like all functions have an inverse, trigonometric functions also have the inverse.



Trigonometric functions – Angles gives ratio of sides;
Inverse trigonometric functions – Ratio of sides give angles

An illustration of trigonometric functions and inverse functions
What is the measure of ∠L?



$$\text{Tan(L)} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{35}{65}$$

Strictly speaking, we have to find the measure of an angle given the ratio of the length of sides. The trigonometric functions – sin, cos, and tan – give us the ratio of the length of sides, given one of the acute angles of right triangles. But the need of the question is just the opposite – we have to find the angle given the ratio of the sides. This is a basic and classical case of using the inverse of a function – using a function to do the opposite of what it is made to do! Thus, if we use the inverse of the tan function, we will get to use it to find the angle for a given ratio of sides! We use an inverse trigonometric function here!

$$\angle L = \tan^{-1}\frac{35}{65}$$

Third scenario
In case the known is a mix of angles and sides, for example, two sides and one angle of a triangle are known, the computations do not change except for more arithmetical steps. The above two scenarios still hold true and adequate.

Summing up

The sine, cosine, and tangent (abbreviated as sin, cos, and tan) are three primary trigonometric functions, which relate the angle of a right-angled triangle to the ratios of two sides' length.

The sec, cosec and cot are the multiplicative inverse of the primary functions, respectively; what it implies is that sec, cosec, and cot are the arithmetical reciprocals of their respective primary function's ratios of sides. The inverse trigonometric functions do the exact opposite of the functions – they take the ratio of sides as input and give the measure of the corresponding angle.

An introduction to 'Mathematised Calculus'

Calculus is real-world mathematics, far more than counting, numbers, arithmetic, (Euclidian) geometry, algebra, etc.; do not let your mind revolt against the statement, for example, when we count four apples, it does not mean the weight of the four apples are exactly alike (the weight of any one apple is approximate for the three). It is the most intuitive of all mathematical objects and concepts. No child can be struggling with visualising and verbalising calculus. Calculus education is all wrong; for the best of 'K-12 toppers', it starts and ends with limit and continuity.

A highly practical, real-world, and intuitive understanding of calculus is what we call 'mathematised calculus,' and that is what should be the content of calculus education in school years. Pertinently, it also best handholds us through the methods of calculus – mathematised calculus is the usual arithmetic, algebra, and trigonometry once past the foundational ideas and principles of calculus.

At its heart, calculus is about a world of derived quantities, the derivatives. There are many physical, real, critical entities, such as speed (and velocity), acceleration, electric current, power, electromagnetic force (comes alive due to magnetic flux), chemical product formation cycle, marginal cost, and utility, etc. that are not directly/physically measurable. For example, power is derived out of energy/work capacity and current is derived out of the amount of electric charge flow in a circuit in a given time; more importantly, both power and current are 'independently' meaningful and important quantities.

Expectedly, calculus is also about the opposite – the anti-derivative (integral), undoing a derivation process to get the quantity that was used to get the derivative. For example, the average velocity over a period is the anti-derivative of acceleration (itself a derivative of velocity) and volume is the anti-derivative of area. Similarly, derived quantities could be used to derive another quantity – the double derivative. For example, acceleration is the derivative of velocity, the latter is the derivative of distance travelled (over a period.)

Derived quantities (derivatives) originate in change, detail the change. All the aforementioned derived quantities, and all the others, have one thing in common – they are the rate of change of a 'changing quantity'! For a (continuously) changing object/situation, its rate of change is the real deal, the determinant of many things that matter about that change. For example, speed (at various instants of time) is the rate of change of distance traversed and determines the impact of accidents, the possibility of skidding at a sharp turn, etc.

Calculus is about measuring change; to be precise, measuring the change as it occurs – the change at different instants in relative terms (with respect to time or any variable quantity) to make more sense of the change. For example, knowing the amount of distance travelled is of little value until it is relatable to the time period of that travel.

Anti-derivative describes the effect of change, not details of change. It is the opposite – not the change at any instant, but the cumulative of the instant changes over a period of time (or any other variable quantity.) It is like a sum of the different 'instant, or infinitesimal' values.

What is the nature of changing values? Relevantly, for a changing quantity, an (infinite) series of the actual values of the changed quantities would need to be measured to understand it. But, a series of such numbers will be mathematically unwieldy and yet incomplete with respect to recording the changing quantities (we will soon exemplify this.) Changing quantities are precisely and comprehensively expressed using mathematical entities called functions, without explicitly listing every individual value. Briefly, functions are like input-output converters, quantifying a certain set of output for a set of input quantities.

Every 'uniquely varying quantity' is a 'unique function'. Every changing quantity is expressed as a function. Thus, we find derivative and anti-derivative of functions.

Functions – The mathematical innovation to capture all instantaneous values

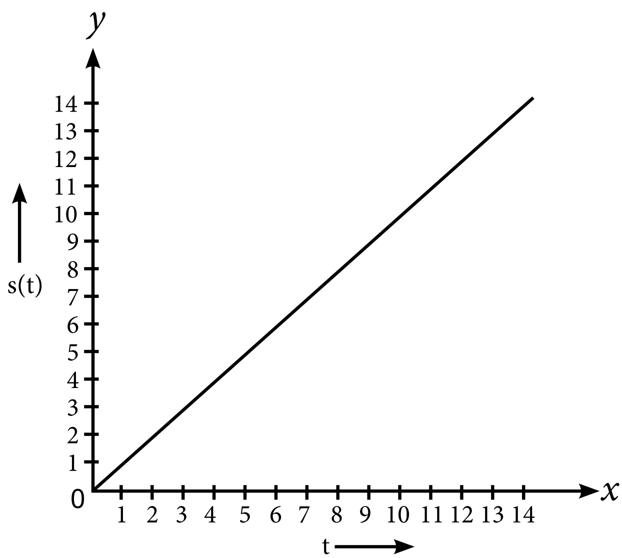
Imagine a bike whose speed at every one second interval is as under:

Time (sec)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Speed (m/sec)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

There are three obvious challenges with this instant values of motion:

- No pattern is directly visualisable (we need to graph it to really see the pattern in motion).
- There is no way to know the speed of the bike at any time other than given, for example, the speed of the bike at 3.5 seconds.
- To know more about the motion, tedious mathematical operations would be needed; for example, to know the nature of the acceleration of the bike, the acceleration values have to be computed for all 14 pairs of speed (start to the first second, first second to the second second, etc.).

However, we can overcome all the aforementioned challenges if we 'summarise' the speed and time relationship through a function. And that function, in this case, is 'speed = time', $s(t) = t$ (speed as a function of time is such that its magnitude is same as the magnitude of time itself, at all times between the start and the fourteenth second); it is more commonly written as $f(x) = x$.



Speed-time graph

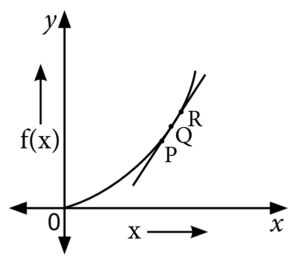
The entire world of mathematical measurement of change is founded on functions. The function gives the (infinite) series of instantaneous values.

When dealing with continuous quantities, functions are the primary mathematical tool for their representation because they can describe how these quantities change continuously. Continuous functions provide a powerful framework for modelling, analysing, and making predictions in various fields like engineering, science, business, etc.

Typically, situations quantifiable by counting represent discrete quantities, and those that need to be measured, or derived, and can take on an infinite number of values within a given range, are the continuous quantities. So, distance and time are continuous quantities whereas the number of students in a classroom is a discrete quantity.

However, functions can also be used for discrete quantification. Planning and controlling the efficiency of the production of limited-size batches of something is an example of creating functions for each discrete situation. For example, the number of batches produced, the number of items in each batch, the number of machines used, and the number of workers are typically whole numbers. Continuous functions need real numbers to be quantified.

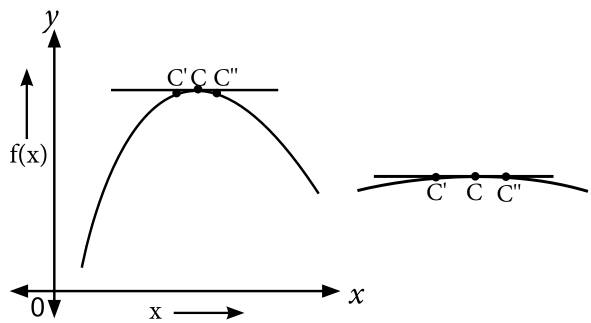
How do we compute the rate of change to find derivatives? The rate of change of a function at an instant, or condition, is the slope of its graph at that instant. Let us not forget that the rate of change varies in a changing quantity, thus, we think in terms of the rate at a point on the graph.



The slope of the tangent PR at point Q is the rate of change of the curve/function at Q

Welcome the idea of limit! Using the idea of limit, we make the slope of the tangent at a point becomes the best approximate value of the slope of the function at that particular point. Limit is the 'science' of infinitesimal quantities, a conceptual breakthrough in mathematics that laid the foundation of calculus – limit allows us to consider an infinitesimal part of the graph around a point that almost overlaps with a straight line tangent at that point. Recall, the slope of a tangent around a point is obtained by the simple rate formula.

Consider a horizontal tangent at point C and points C' and C'' close to C.



The figure on the right is an 'infinitely enlarged' view of an infinitesimal part of the left curve

As the points C' and C'' get closer and closer to the point of interest C, the line becomes smaller and smaller while the slope of the line changes. When the points C', C, and C'' are closest possible, the line becomes a tangent and the steepness (slope) of this line gives the best approximate value of the slope of the curve at the point of interest (point C). Another way to look at it is as follows.

Guaranteeing that limit does not go wrong! The concept of continuity complements the advantages of limit, by ensuring that the chosen infinitesimal part of the graph around the point to find the rate of change does actually represent the slope of the function at that point. Any sharp variation in the slope of the graph at that point is detected as a lack of continuity of the same slope at that point. In such situations, the limit of the function is said to not exist at that point, i.e., the slopes of the function just before and after the point are not the same.

What is the greatest deal about calculus? The derivative and anti-derivative of a function are the same for all the valid inputs for the function (the domain of the function). For example, the derivative of $f(x) = x^2$ is 2x, and it implies that the derivative at the point x = 2 is 4, and the derivative at the point x = 8 is 16.

There is more – solving calculus questions does boil down to knowing or computing the derivative or anti-derivative of the individual terms in the question and then following the usual simplification of the expression much like solving algebraic or trigonometric questions.

We are ready to consolidate this introduction of derivative, anti-derivative, limit and continuity as we read ahead the 'mathematised calculus' chapter.

A note on calculus and 2023!

The idea, joy, and applications get completely lost in calculus education due to its singular and rigorous rooting in the idea and computations, limit and continuity. 2023 is an -interesting milestone in calculus, and its education. In 1823, the French mathematician, Augustin-Louis Cauchy presented the text Résumé des leçons sur le calcul infinitesimal ('Summary of Lectures on the Infinitesimal Calculus,'), his first book devoted to calculus, originally written to benefit his École Polytechnique students in Paris. The book is a remarkable work of conceptual vision and laid the foundations of the rigourised, formalised, particularised, and proceduralised foundations, concepts, and practice of using calculus.

The book had a sweeping effect on mathematics as a whole, and it massively guided and accelerated the development of 'abstracted and methodised mathematics.' However, it was meant to popularise and strengthen the correct applications of calculus among engineers and scientists. It was not meant to be used in introductory calculus education in schools, but that is exactly what happened, and an intuitive understanding of calculus was lost.

Mathematised Calculus

Change is the only constant

Change is an inherent and unchanging reality of our world. Change refers to any alteration, modification, or transformation in the conditions of an object or situation (system). It can occur gradually and steadily over time without distinct breaks or interruptions or can be abrupt and sudden. Erosion and weathering of rocks, adaptation, and evolution of living things are changes that are very slow while volcanic eruption, earthquake, and landslide are examples of sudden changes. Continents, which seem fixed and immovable, are actually in continuous motion – a few centimetres per year. Middle school physics is built on the notion of constant acceleration (recall, F = m.a), but constant acceleration is a myth (even in deep space travel).

Let us not be deceived by many things around us that seem stable or unchanging, it is only a simplification of reality to make it easier to understand and compute at a preliminary, best approximate, and conceptually correct level. For instance, when we talk of averages, such as average speed over a 5-hour journey, we do not mean that the average speed was even momentarily an actual speed, it is just one good approximation of a range around which the speed lay in those 5 hours. The actual speed at different instants was not a constant that the average speed is computed to be.

The undeniable truth is that everything is in a continuous state of flux, change is the best hallmark of how our world is. To precisely and comprehensively understand our world, we need to explain and also measure the way change becomes evident in all things around us.

To understand the world, we need to understand the change

Fortunately, our world can be visibly categorised into two broad kinds of objects –

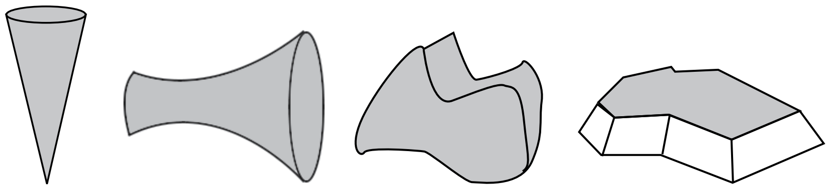
1. Objects that are stationary (buildings, trees, books, …)
2. Objects that move, or are in motion (motion may be the most ancient human fascination, starting with the motion of celestial objects)

Importantly, it is natural to think of change in stationary objects only through the lens of change in their position. But this would be missing the point about the change in stationary objects – change in such objects may also be in their weight, dimensional measures, the composition of matter in them, and others (to name just the quantitative descriptors).

Thus, the first characteristic of a change in an object is any kind of measurable difference in it between two instants of time, or any other conditions (such as in response to a change in pressure, temperature, etc.)

One of the special measurable differences of this kind is also the change in the dimensional measures – surface area, and volume (space occupied by a thing) of objects. For instance, imagine a rectangular packet of tea leaves tearing apart and a heap of the same tea leaves forming on the ground; the surface area of the heap and the packet would have changed (not the volume). And the usual geometric computations would not help in finding the surface area of the heap.

Similar computational challenges abound when attempting to find surface areas and the volume of 'curved objects', such as the following; (Euclidian) geometry does not work for curved surfaces and objects.

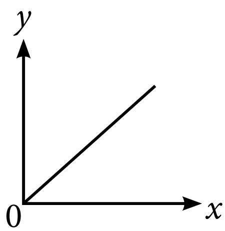


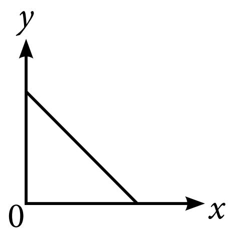
As to things that move, or are in motion, it is intuitive to think of the motion itself as representing change. Indeed, it is – change of position/place of the object in motion – but that is what motion is inherently about; there is no motion unless and until there is a change of position involved in it. However, a steady motion wherein the distance travelled, the time taken for that travel, and the direction are the same (if the direction is also relevant) over a time period, the motion would not be called to be changing. The motion would be said to be changing only when the direction and/or the distance travelled over the same time period changes.

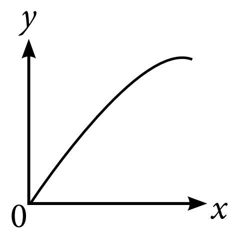
Changing motion is literally the norm across the universe. All celestial bodies move along a curved path, may it be circular, elliptical or parabolic, hyperbolic, etc.; elliptical movements are the most common ones. This implies that the direction of motion of the celestial bodies constantly changes, and also implies that the distance traversed in fixed time intervals also changes constantly (in elliptical and parabolic motions).

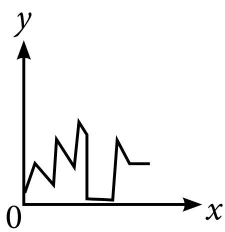
Thus, change occurs IF a relationship between two, or more quantities is NOT steady; for example, if a body is moving in a way that the distance travelled by it over periods of time is different, then the motion is said to be changing.

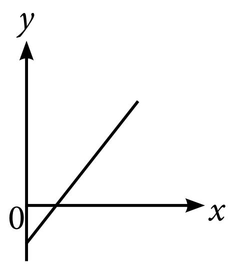
Happily, graphs of the relationship between quantities are a very easy way to identify existence as well as the nature of change between the quantities. Here are a few examples of how graphs can show if change exists in a relationship, and how does the change look like:

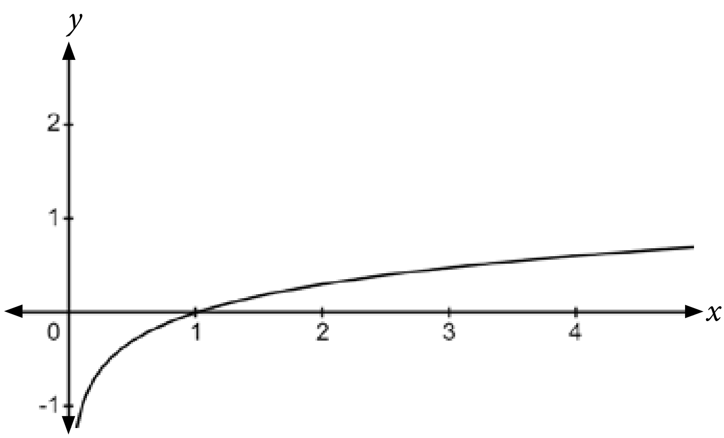

y
0
x

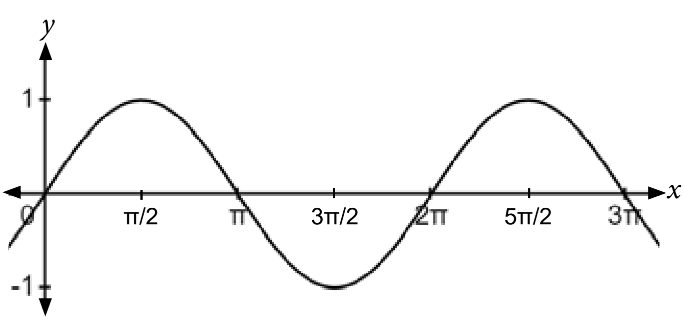

y
0
x

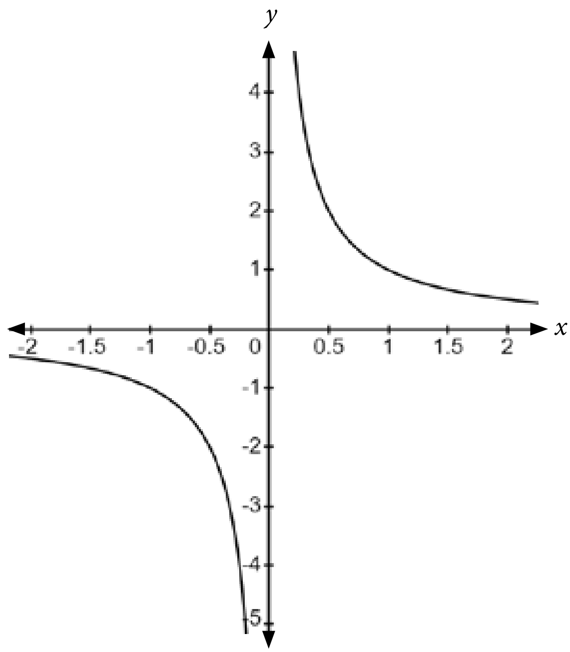

y
0
x

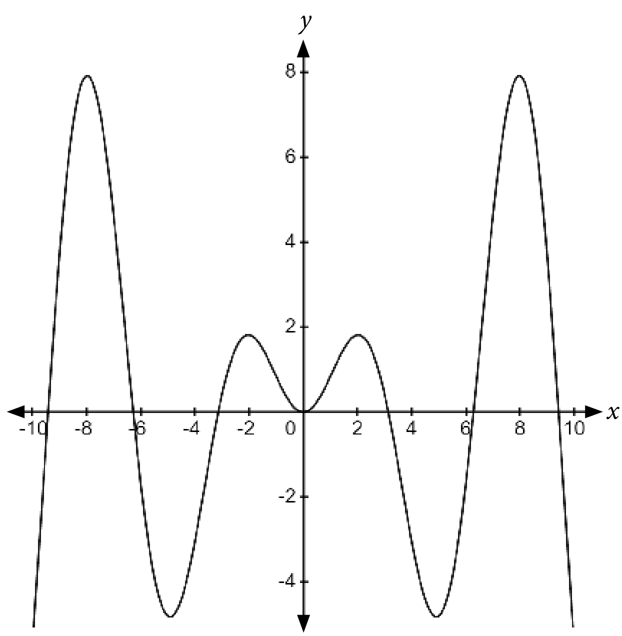

y
0
x


y
0
x


y
2
1
0
1
2
3
4
-1
x


y
1
0
π/2
π
3π/2
2π
5π/2
3π
-1
x


y
4
3
2
1
x
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
-1
-2
-3
-4
-5


y
8
6
4
2
x
-10
-8
-6
-4
-2
0
2
4
6
8
10
-2
-4

Interestingly, change can be registered only when it can be objectively measured quantitatively (for example, change in numerical values, 30°C to 45°C) or qualitatively (for example, change in colour or texture). However, change as a subject of mathematical interest implies changes occur when quantifiable relationships are not steady.

Quantifying change – A series of instant values

It is very interesting to realise that the biggest and quickest of changes are also in 'slow motion', steady, gradual, or what may be called 'a series of unique instantaneous values'. Time is amazingly divisible, and a change that appears in just one-tenth of a second is also slow and steady when looked at time frames that are one-hundredth of a second. The first one-hundredth second of the start of that change will register some kind of quantitative difference, the second one-hundredth second will bring in another quantitative difference (that will add to the quantitative difference of the first one-hundredth second), and so on.

The amount of change that is measured between the zeroth second to the one-hundredth second is the amount of change in the first one-hundredth second, and the amount of change between the end of the first one-hundredth second to the end of the second one-hundredth second is the amount of change in the second one-hundredth second.

At some slicing of time, all changes are just a series of many instantaneous values of change (all adding up together). Thus, to know a change, we need to study it as a series of instantaneous changes; the instants depend on the pace of the change, it could be the amount of change per minute, per second, per millisecond, etc.

However, the idea of instant has very interesting implications for actually measuring change. An instant means now and it is almost in some changes, the magnitudes between the beginning and end of observation could take an infinite number of possible values.

Such changes could be considered as an accumulation of an infinite number of infinitesimal or 'infinitely small' changes, occurring at each moment or instant. Such changes are so small that the change occurring between two precise 'moments' is nearly imperceptible. For example, the change in height of a child between his first and second birthday.

When we consider all the infinitesimal changes at all instants together, it creates the impression of an unbroken, continuous change in a system and gives a complete understanding of the behaviour of the change. When graphed, these changes are represented as a continuous line or curve.

To understand the events that are an accumulation of infinitesimal changes, and can change at any point and in any magnitude requires a language or framework that can effectively describe these dynamic and evolving systems. This mathematical language that represents continuous changes is a function.

Function – Capturing realities in mathematical expressions

Functions are mathematically expressed relationships of real-world situations, and they are such that for each change in any of the variables in the relationship, however small, a change is observed in some other variable of the relationship.

Continuous functions are a fundamental tool for understanding and making predictions about the behaviour of continuously changing systems in a wide range of fields. They enable accurate modelling, analysis, and optimization, making them essential for addressing complex and real-world problems.

To know more about continuous functions, refer to the Note 3 at the end of the chapter.

One of the real-world situations expressed as a function is the relationship between distance (D) and time (t) where both are variables and distance is dependent on time, i.e., D = f(t). Here, 'D is a function of variable t' means that there is a mathematical relationship that describes how D changes or depends on changes in t. So, the function 'f' takes the value of 't' as input and produces the corresponding value of 'D' as output. There is a unique value of D for every value of variable t.

Various mathematical functions define real-world situations, such as $f(x) = x^2$ is a quadratic function that represents a parabolic path, $f(x) = x^2 + 4$ is a quadratic function with a vertical shift such as energy levels or distances with a constant offset, $f(x) = \frac{1}{x}$ is a rational function that describes situations where one quantity is inversely proportional to another, f(x) = sin x is a trigonometric function representing a sine wave that models oscillatory behaviour, etc. It is written in a way that one quantity is seen as varying, or dependent on the way other quantity (independent) vary. For example, in the function, y = sin x, where x is the independent quantity and as x varies the value of y (dependent quantity) varies. The notation commonly used to represent to describe functions is either y or f(x).

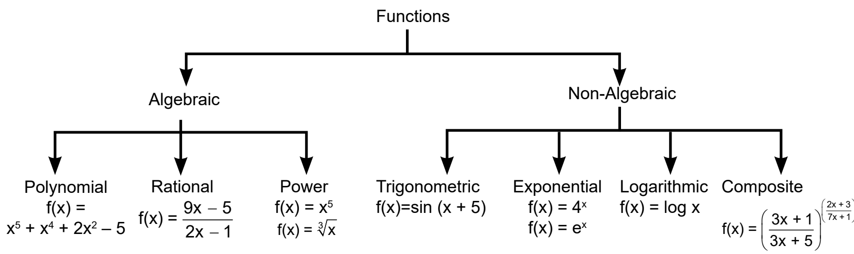
Functions as input and output processors

Input is a quantity that is 'entered' into a function. The quantity should be such that it is valid for the function, for example, the function $f(x) = \sqrt{x}$ is only valid for positive values of x. And, after processing, f(x) returns a value, the output. For example, if x = 4 then the output f(x) is +2 and –2.

The possible set of valid values of the 'input,' the independent variable, is called the domain of the function. The processed set of values of the output, the dependent variable, is called the range of the function.

Finite set of functions

Function $f(x) = x^2$ represents a parabolic function and function g(x) = ax + b represents a straight-line, both being algebraic functions. However, h(x) = sin x, expresses a sinusoidal wave and is a trigonometric non-algebraic function. We can categorise functions more elaborately as follows:



Different categories of functions

Instantaneous value of a function

Instantaneous value is essentially the value of the function at a specific point in time or space, taken at an infinitesimally small moment. For modelling and analysing various natural phenomena and real-world systems, we need to quantify the change in instantaneous values of the function.

The challenge in computing change in instantaneous values

The challenge in finding the change in instantaneous values or rate of something is the measurability of changes at that particular 'instant, or point/condition (for example, measuring distance travelled at an instant, i.e., measuring the distance covered for a duration that is nearly zero). The divisor in such computation of rate is nearly zero (we call these nearly zero quantities 'infinitesimal', which means infinitely small). Mathematically, such quantities/ numbers can be visualised but any attempt to physically measure such changes is near impossible; imagine measuring the distance travelled by car in 0.001 seconds (0.001 seconds being the time assumed to represent 'an instant').

Thus, there are three challenges which we face while computing the change in instantaneous value or rate of something:

- The physical challenge to precisely measure a small quantity within a short time frame.
- The computational challenge in which the divisor is almost zero.
- The conceptual challenge that such small quantities do exist.

Solving the physical challenge of instantaneous values

Physically, it is impossible to correctly measure a quantity which is small in magnitude for a small measurement window. This physical challenge is resolved by using the idea of an indirect quantity, a derived quantity.

A derived quantity is a new, special quantity derived from another quantity (primary quantity). It is the quantification of some new aspect of a change in primary quantity. It is not a directly measurable quantity. It can only be computed using primary quantities.

Here are some examples of the derived quantities – power is the quantity derived from the primary quantity energy/work; force is derived from momentum; electric current is derived from electric charge; and electromagnetic force is derived from flux.

The derived quantity is called the derivative of the primary quantity out of which it is created.

Derived quantities out of function

Functions represent the world of relationships among quantities, and they are also the source of deriving new quantities or information from the primary quantities. This derived quantity or the new function obtained from the primary or original function is the derivative of that function.

The derivative of a function is the mathematical operation that works on a function to 'derive' indirect meaning(s) from the function. The primary meaning of a function lies in the relationships it represents between quantities; for instance, interest amount is the direct meaning derived out of the function that relates interest earned on a principal amount deposited in a bank for a time. From the derivative of this function, we can obtain indirect information, such as the instantaneous rate at which the interest is accumulating etc.

Derivative quantifies some new aspect of a changing quantity, for example, when we know the relationship between time and distance traversed by an object in motion, we can derive the speed and acceleration of the object in motion. The derivative represents the rate of change of distance (the 'something else') with respect to time. Evaluating the derivative at a specific instant gives us the instantaneous speed (the 'something') of the object at that moment. To be precise derivative gives a very specific new knowledge about something in change – the instantaneous value of 'something' with respect to the change in 'something else'.

The conceptualisation of the derivative of a function can be visualised as detailed hereunder:

- Something is continuously changing (slow, fast, regular or irregular ...); for example, a car in motion continuously changes its position. It can change its position at a fast pace when on a highway or at a slow pace when in a traffic jam. Whatever the case is, it is continuously changing its position.
- The change in position is measured by a (physically measurable) quantity. Distance is that quantity which measures the change in position of any object in motion.
- The quantity that reflects the change in motion is distance. It indicates the change happening for any moving object.
- The rate of change of that quantity could also be changing. The change in distance in the various units of time could also be a variable, changing; for example, more distance is traversed in a certain period, as compared to an equal another period.
- The instantaneous value of the rate of change represents 'something else' which is another quantity related to the original quantity being measured. The rate of change of distance with respect to time represents speed, which may also change.
- The value of 'something else' is the value derived out of another quantity. Speed, which is the rate of change of distance with respect to time, is derived from distance.
- The derived quantity is called a derivative. Speed is the derivative of distance.
- The derived quantity is not a direct/primary measurable observation. Speed cannot be directly measured; in the way distance and time are measured.

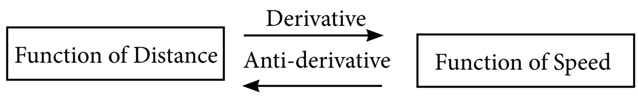
- The derived quantity changes in tandem with the change in rate of change of the primary quantity. As the distance (which is the primary quantity here) varies over the different time periods of motion, the speed (which is the derived quantity out of distance) also varies.
- Theoretically, a derived quantity is also a function which can change with respect to any other quantity. It must be possible for the derived quantities to derive another quantity. The rate of change of velocity/speed is acceleration. Acceleration is thus derived out of velocity, which is in itself a derived quantity out of the changing distance.

Mathematical expressions using derivatives (Differential equations)
Recall that algebraic expressions are combinations of constants and variables that are put together using mathematical symbols, and algebraic equations are expressions that are set equal to zero. It is interesting to think that equations can also have expressions that incorporate changing conditions quantified through the rate of change. Such expressions are common, we mathematically express them every day and when used under scientific conditions, for now, they are called derivative equations or differential equations.

Wherever there are changing quantities in the 'equation' of a thing, the situation is mathematically expressed as differential equations. These equations can be used to configure everyday life to rocket science.
Refer to Note 4 to learn more about the differential equations.

Anti-derivative of function
As the name suggests, it is mathematically the opposite of the idea and the operation of the derivative. Let us construct the understanding of anti-derivative – one dimension at a time, out of the definition of derivative.

We know that a derivative is a function that gets created from another function. Thus, the anti-derivative must also be a function created by another function; (both 'input and output' of the derivative is a function, so the reverse view of derivative will also be 'input and output' as functions). For example, the function of speed gives the function of motion itself, the function of distance (motion is about a change of positions, distance), to be precise.



Next, we know that a derivative is a rate. The implication of being a rate is that it is a slice of the action, a 'part of a whole.' For example, the speed at an instant in a motion. What may be the anti (or opposite) of a 'part of a whole'? 'The whole' itself. For instance, what may be the anti-derivative of the (derivate) speed?

Let us take a second to scrutinise speed; it is the distance travelled in a unit of time (whatever be it), and it is a part of the (total) distance travelled. Indeed, a speed of 54 m/s implies that each slice of 1 second of motion is a distance of 54 m. The anti-derivative of speed is, in fact, distance, 'the whole'.

Of course, in the story of anti-derivative, 'the whole' needs to be identified because 'a whole' could also mean the universe. 'A whole' as anti-derivative needs to be specifically limited, and that is what is next explored.

We also know that a derivative is the value of something at an instant. What may be the opposite of an instant? A period of time, an interval of time. Indeed, anti-derivative qualifies (i.e., further explains) the quantity discussed previously (total distance travelled). The opposite of an instant is accumulated time or a period of time. And to the extent that anti-derivatives operate over a specified range of values of the changing quantity, the specification of that range is an input in finding anti-derivatives. Thus, the anti-derivative quantifies how much something has changed over a time period.

Anti-derivative is kind of a difference between two quantities – one at the start of the period or a state of things, and the other at the end of the period or state of things. Another way to see it is the summation of the changes or cumulative changes over a period or the range of change of another thing.

Common examples of the application of anti-derivatives to find out the amount of change in something (that is changing) are – from the anti-derivative of the function that represents the rate at which the tumour grows over time, the accumulated change in the size of the tumour inside the body of animals can be obtained. The anti-derivative of the function that represents the rate of population growth over time can be used to find the total population increase over that time interval. The impact of a head-on collision of two cars is a series of (very fast happening but) small changes in position after the collision. The anti-derivative of the function that represents the positions of the crashing parts of the cars can be used to provide the cumulative change or the displacement of the cars during the collision.

Here are a few more examples of the deployment of the idea and operation that is anti-derivative:

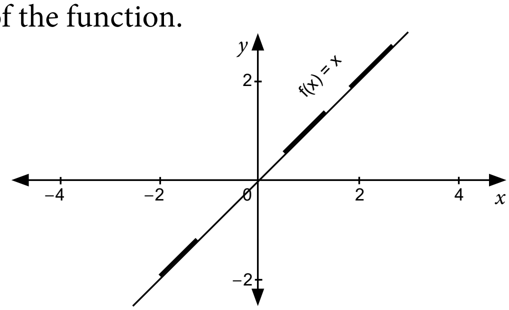
Average/ Function	Anti-derivative	Explanation
Function	Average value of a function	Average value of a function over a range is the anti-derivative of the function.
Area	Volume	Volume is the anti-derivative of the area; over a dimension; similarly, area is the anti-derivative of one dimension.
Density function	Mass	Mass of an object is the anti-derivative of a density function (which is mass per unit volume) over a given volume.

Let us find the derivatives and anti-derivatives of functions through their graphs. But before this, read about slopes from Note 1 and Note 2 given at the end of the chapter.

Derivative of f(x) = x

The derivative of a function at a point is the rate of change of the function at that point. Graphically, the rate of change of a function at a point is the slope of the curve at that point. A study of the slope of a curve indicates the derivatives of function at various points.

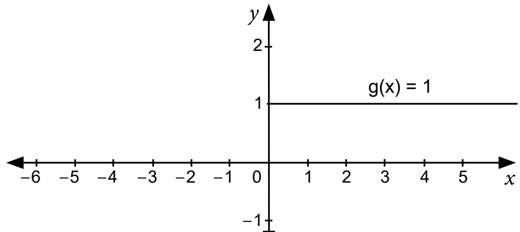
Consider f(x) = y = x which represents a linear function. The derivative/slope at each point within a domain of the function (the possible values of the input value x) having a linear graph is constant, that is, y changes at the same rate or constant value within the domain of the function.



Various tangents on the linear graph of f(x) = x

Since the slope is always a constant value for a linear function, the derivative of a linear function is a constant function and can be represented mathematically as g(x) = c, (c is any constant) and graphically as a horizontal line parallel to the x-axis.

For linear function f(x) = x, the slope is 1. Therefore, the derivative of f(x) is g(x) = 1, the slope of the given function f(x) = x, which is a constant function representing a horizontal line parallel to the x-axis. We will derive it mathematically later.



Graphical representation of derivative g(x) of f(x) = x

Since, the graphs of all linear functions are derived from the graph of y = x and possess the same properties, the slope/derivative of all linear functions is a constant.

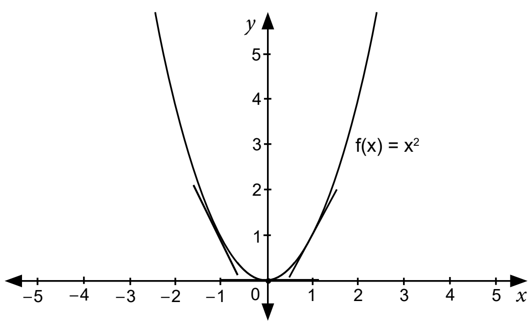
Derivative of f(x) = x²

It is convenient to study the derivative of any function through its graph, and we will work with the graph of the given function $f(x) = x^2$ to find its derivative.

Recall, the derivative of a function is also a function and in its graph, the x-coordinate is the same as the function's x-coordinate and the y-coordinate is the value of the slope of the given function.

Thus, for the given function, we need to study the behaviour of the slope at different points of its graph to get the corresponding points to plot the graph of its derivative function.

Let us consider the graph of the given function $f(x) = x^2$, along with three tangents drawn at different points, to discuss the nature of the slope of the function.



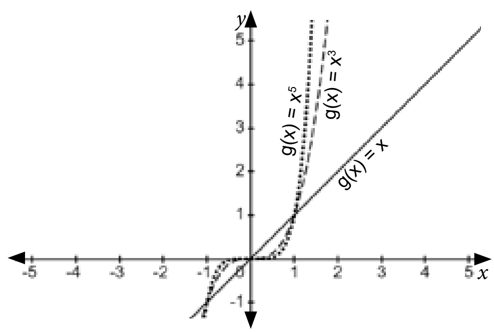
Graphical representation of $f(x) = x^2$

We can observe the following in the graph of $f(x) = x^2$:

- For negative values of x, the slope of the given function is negative and for positive values of x, it is positive.
- The slope is zero at x = 0, where the tangent is horizontal to the x-axis, in fact, it coincides with the x-axis.

- Based on these observations, two imperatives emerge for the derivative function of the given function, $f(x) = x^2$, where the derivative function's value is the slope of the given function (whose derivative is supposed to be explored).
- The derivative function passes through the origin, since at $x = 0$ the slope is 0.
- The values of slopes are increasing for $x > 0$ and decreasing for $x < 0$.

This suggests using the above arguments, the possible derivative function graphs could be any of the following $y = x$, $y = x^3$, and $y = x^5$.



Graphical representation of possible derivatives g(x) of f(x) = x^2

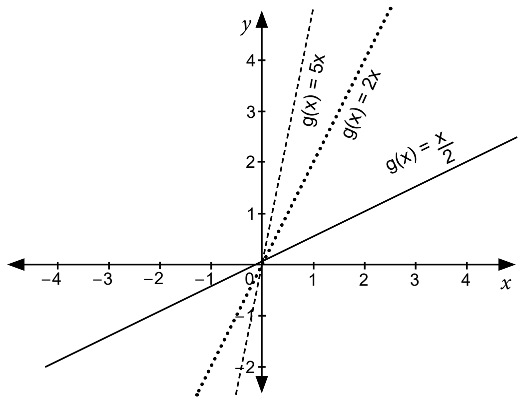
However, we also observe a gradual decrease/increase in the graph. This can be affirmed by the values which the function takes when the values of x are put in the function $f(x) = x^2$.

When $x = 0.5$,	$f(x) = 0.25$
When $x = 1$,	$f(x) = 1$
When $x = 1.1$,	$f(x) = 1.21$ and so on.

Also, there are no sudden dips/rises in the values of the slope of the tangents considered in the graph of the function $f(x) = x^2$ because it is a continuous function. Recall that the steeper the tangent, the more is its value of slope. This suggests that the values of the derivative function would not be large for a small value of input. That is, the derivative function cannot be a curve such as x^3 or x^5, where for a small value of x we have a large value of the function.

This suggests a straight line as a derivative for x^2.

There can be many functions with this possibility that can be seen in the graph.

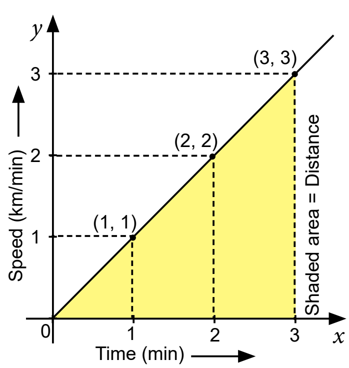


Graphical representation of possible derivatives g(x) of f(x) = x^2

The takeaway from this derivative graph is that the derivative of all quadratic functions is a linear function. We have logically derived the nature of the derivative function of a quadratic function. Later we will explore which of the above graphs is the actual derivative graph of the given function.

The derivative of x^2 is 2x. For now, let us generalise that the derivative of all quadratic functions is a linear function.

Graphically exploring the anti-derivative of speed



Speed-time graph

The graph represents a car in motion having a linear speed or we can say that the speed increases linearly with respect to time, i.e., at t = 1 min, it has a speed of 1 km/min; at t = 2 min, it has a speed of 2 km/min.

Since distance = speed × time, it can be best interpreted by the shaded region. The anti-derivative of speed is the area of the shaded region, which is the distance covered by the car in motion.

In fact, the anti-derivative of a function is a quantity that is the area under the curve of the function.

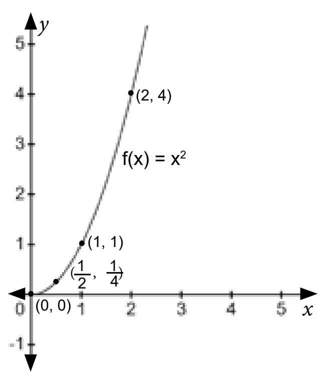
Anti-derivative of $f(x) = x^2$

We aim to find a function whose derivative is x^2. The anti-derivative function is one whose range values are the same as the various values of the area under the curve $y = x^2$ in some interval.

Let us assume the interval [0, 2]. The graph of x^2 in the interval [0, 2] can be obtained by making a table of the various points of x and the corresponding points of y.

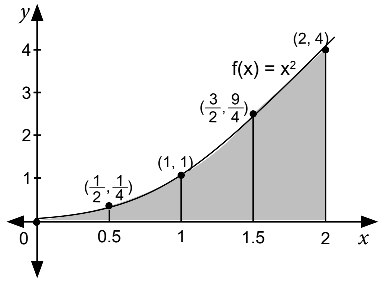
x	0	$\frac{1}{2}$	1	2
y	0	$\frac{1}{4}$	1	4

And, it can be shown as under.



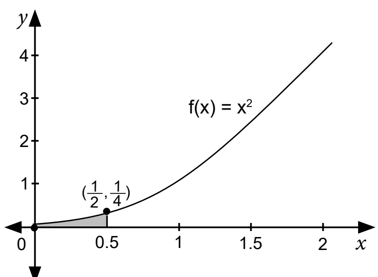
Graphical representation of $f(x) = x^2$ in the interval [0, 2]

Arithmetically, it is tough to find the exact area of the region that is curved. However, finding its approximate area is always possible and that basically serves our current purpose of broadly finding the nature of the function that is the anti-derivative of x^2. Logically, breaking the intervals [0, 2] into small sub-intervals would make better sense.

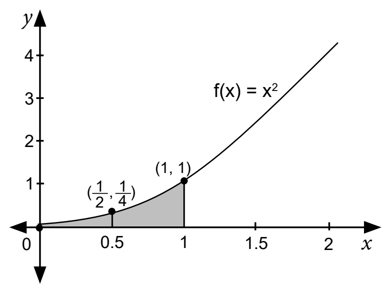


Graphical representation of the area under the curve of $f(x) = x^2$ in the interval [0, 2]

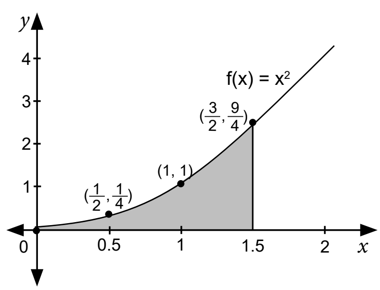
The area under the curve in the interval [0, x] can be approximated by a triangle with base x and height x^2.



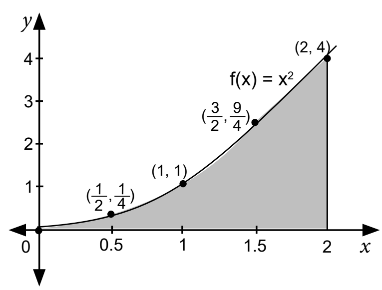
(a) Area between interval $\left[0, \frac{1}{2}\right]$



(b) Area between interval [0, 1]



(c) Area between interval $\left[0, \frac{3}{2}\right]$



(d) Area between interval [0, 2]

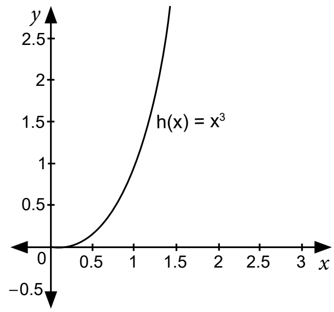
Area in the interval [0, x]	Area of triangle $= \frac{1}{2} \times$ base $\times$ height			Coordinate for anti-derivative graph (x, area in the interval [0, x])
$\left[0, \frac{1}{2}\right]$	$\Delta 1 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4}$	$= \frac{\left(\frac{1}{2}\right)^3}{2}$	$= \frac{1}{16}$	$\left(\frac{1}{2}, \frac{1}{16}\right)$
[0, 1]	$\Delta 2 = \frac{1}{2} \times 1 \times 1$	$= \frac{(1)^3}{2}$	$= \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
$\left[0, \frac{3}{2}\right]$	$\Delta 3 = \frac{1}{2} \times \frac{3}{2} \times \frac{9}{4}$	$= \frac{\left(\frac{3}{2}\right)^3}{2}$	$= \frac{27}{16}$	$\left(\frac{3}{2}, \frac{27}{16}\right)$
[0, 2]	$\Delta 4 = \frac{1}{2} \times 2 \times 4$	$= \frac{(2)^3}{2}$	= 4	(2, 4)

From the above calculations, we can deduce that the approximate area under the curve $f(x) = y = x^2$ corresponding to any interval [0, x] is $\frac{x^3}{2}$.

On plotting the coordinates for the anti-derivative graph, we will get an approximate graph of the anti-derivative function of x^2, $h(x) = \frac{x^3}{2}$.

This is an approximate value, the real value will be less because the curve of the graph is concave. That is, instead of the anti-derivative being $\frac{x^3}{2}$, it can be $\frac{x^3}{c}$, where c is a real number.

For conceptual exploration, we can ignore 'c', and we can say that the anti-derivative function of $f(x) = x^2$ is $h(x) = x^3$. The graph of this is as under.



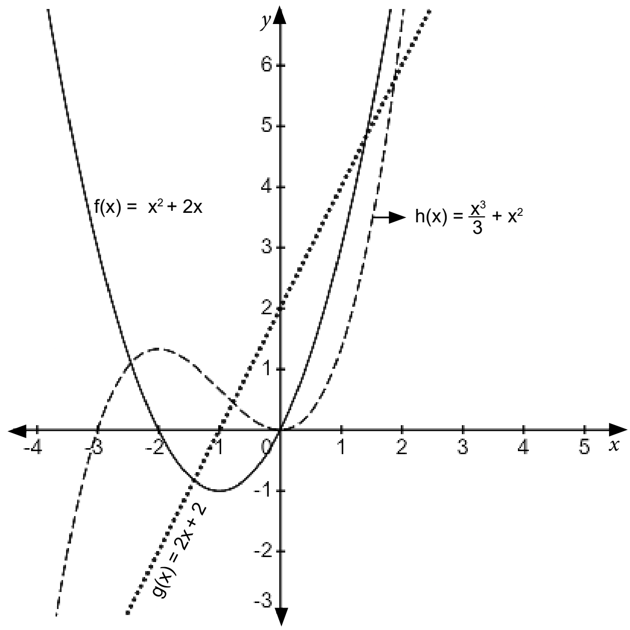
Graphical representation of anti-derivative h(x) of f(x) = x^2 ignoring the constant c

Thus, the anti-derivative of any quadratic function of the type $ax^2 + bx + c$ would be a three degree function. On actual mathematical computation, the anti-derivative of x^2 is $\frac{x^3}{3}$.

On combining the result for the linear and quadratic functions, the derivative of $y = x^2 + 2x$ is the sum of the derivatives of x^2 and 2x. The derivative of x^2 is 2x and that of 2x is 2. Thus, the derivative of $y = x^2 + 2x$ is $2x + 2$.

The anti-derivative of $y = x^2 + 2x$ is the sum of the anti-derivative of x^2 and 2x. The anti-derivative of x^2 is $\frac{x^3}{3}$ and that of 2x is $\frac{2x^2}{2} = x^2$.

Thus, the anti-derivative of $y = x^2 + 2x$ is $\frac{x^3}{3} + x^2$.



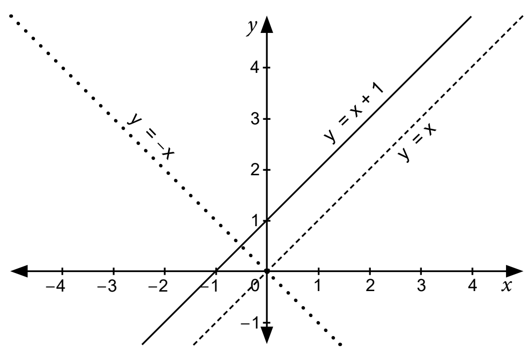
Graphical representation of function $f(x) = x^2 + 2x$ and its derivative $g(x)$ and anti-derivative $h(x)$

Now, we can find the derivative and anti-derivative of any function through their graphs. But there are infinite functions for infinite realities. Fortunately, math is the language to be used for expressing patterned conditions and relationships. And, inexplicably, it just so happens that just a few tens of patterns (i.e., well-defined, repetitive, 'universal' behaviour) lie at the core of the infinite realities. Expectedly a core set of functions – parent functions – do nearly express all kinds of situations.

The parent functions

When we graphically represent functions we can see that many functions' graphs look alike and follow similar patterns because these functions share the same parent functions. Parent functions are basic and the simplest form of functions. These functions serve as fundamental building blocks for constructing more complex functions. The complex functions from the same family of parent function can be easily recognised/graphed bearing the marked features of the parent function. Conversely, by taking the parent function's graph through various shifts, flips or stretches, all the functions within a family of functions can be derived.

There are infinite possibilities for creating a unique function from just one parent function. For example, $y = x$, $y = -x$, and $y = x + 1$ all represent a family of straight lines that can be seen in the given graph.

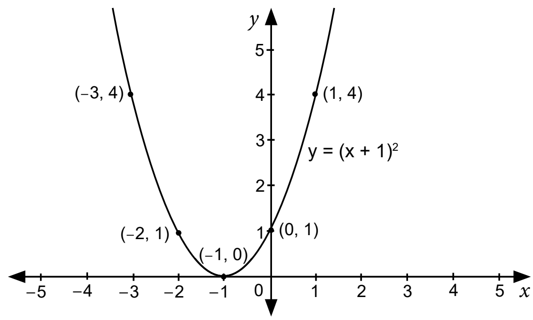


Graphical representation of a family of straight lines

If we observe the graph carefully, we will notice that the graph of $y = x + 1$ is shifted up by 1 unit from $y = x$ and both have the same shape of the graph. Similarly, $y = -x$ is a reflection of $y = x$ about the y-axis. However, both the functions, $y = x + 1$ and $y = -x$ look similar in a definite way. The transformations of the parent function in no way change the shape of the parent graph, and follow the basic characteristics of the family as defined by the parent function.

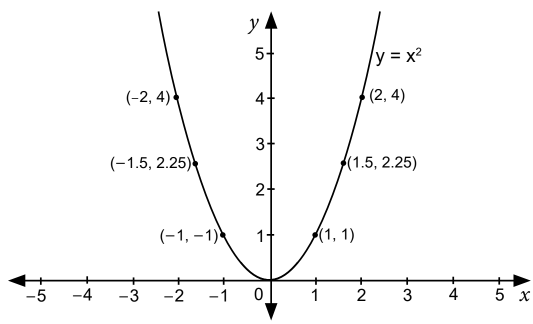
As another example, let us draw the graph of $y = (x + 1)^2$. Following the standard graphing technique, we create the following table of coordinates to plot the points on the graph.

x	−3	−2	−1	0	1
y	4	1	0	1	4



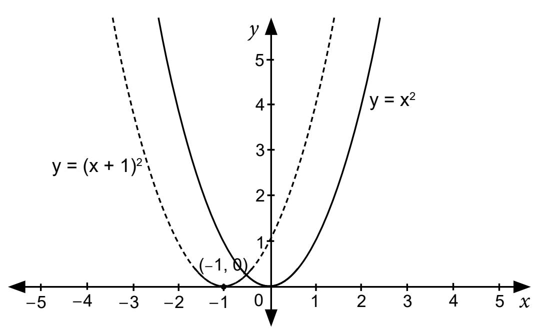
Graphical representation of the function $y = (x + 1)^2$

Let us also create the graph for $y = x^2$ using the same graphing technique.



Graphical representation of the function $y = x^2$

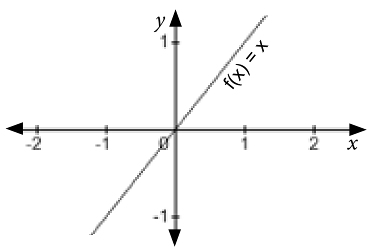
When the above two graphs are superimposed onto a single graph, they would look like:



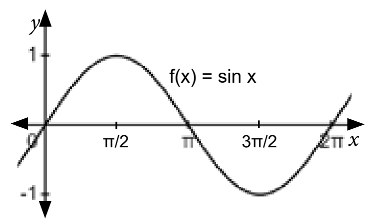
If we observe carefully, $y = (x + 1)^2$ is a parabola with its vertex at (−1, 0). Its graph is similar to that of $y = x^2$, a parabola, with vertex at (0, 0). Thus, knowing the graph and properties of $y = x^2$ can help us to know the graph and properties of the function $y = (x + 1)^2$ as well. Hence, $y = x^2$ is called the parent function of all other degree 2 polynomials, such as $y = (x - 1)^2$ and $y = (3x - 4)^2$. The graphs of both of these functions have the same shape, however, the vertex the case of $y = (x - 1)^2$ is (1, 0) and that for $y = (3x - 4)^2$ is $(\frac{4}{3}, 0)$.

We can combine the graphs of these parent functions to create a new combined function. Let us explain this with the help of a combined function f(x) = xsin x.

This function is a product of the linear function x and the sine function sin x. First, visualise the graphs of the individual functions.

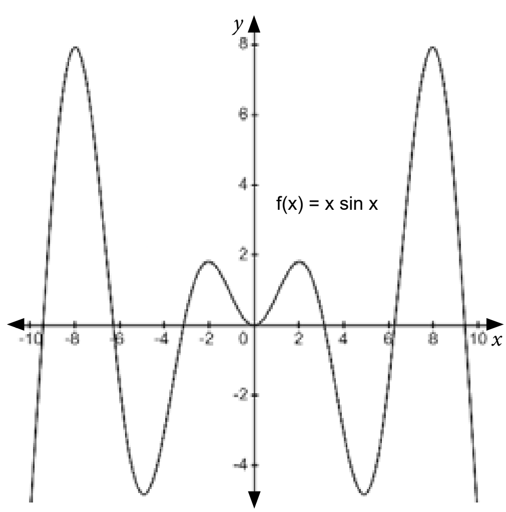


Graphical representation of f(x) = x



Graphical representation of f(x) = sin x

The sine curve oscillates between −1 and 1. However, its product with x will change the amplitude (the maximum height of a wave) of the combination function. This can be verified from the graphical representation of f(x) = x sin x.



Limit solves the computational challenge of instantaneous values

Let us get back to the challenge of finding the instantaneous value of a changing quantity and speed as an example of the same. We know 'the rate of change of distance', is speed, it offers the 'average' of the distance traversed over a period of time. The idea of 'average' is embedded in the idea of rate. The discussion eventually boils down to overcoming the limitation of the mathematical expression of average, which is the best mathematical means possible to find how much something changes due to the change in something else (such as time).

Recall, average speed is conceptually akin to a division expression (average is a kind of ratio that in its standard form is best read as division) and divisor of (almost) zero makes the quotient skewed towards being disproportionately bigger.

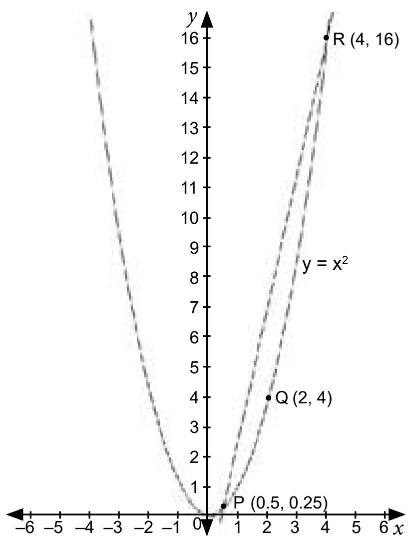
In the discussion on derivatives, we figured out that we can overcome this challenge by using the idea of an indirect quantity, a derived quantity, called the derivative; for example, the instantaneous value of speed (of something in motion), at a point, is derived from the way distance changes over time – the rate of change of distance travelled at that instant.

Conceptually this was a breakthrough, but computationally, finding the rate of change at an instant remained a challenge (when computing speed, instantaneous strictly means 'zero' time duration of the observation, and the distance traversed in that 'zero duration').

For finding instantaneous speed, we cannot have 'zero' time duration a divisor. The next best thing is to make the time duration so small that it is non-zero but tends to be as close to zero as possible. We need a non-zero divisor for the computation of instantaneous change to be possible, but to reflect instantaneous values, the nonzero divisor must be the smallest possible. When the independent quantity is non-zero, yet approaching zero, it is said that 'the limit of the quantity is zero'. This non-zero, but closest possible to zero approach, where the rate of change in the value of x is non-zero but near zero is called the limit of a function f(x).

Now let us study an example of a curve, parabola $y = x^2$, to see how we can get infinitesimal tangent on a point on the curve and use it to find the slope at that point on the curve.
We will find the approximate value of the slope on the point of interest Q (2, 4) by continuously reducing the distance between the two random points P and R (above and below point Q) on the curve to reach the closest to point Q to get the infinitesimal tangent.

The slope of this tangent would give the slope of the curve at point Q.



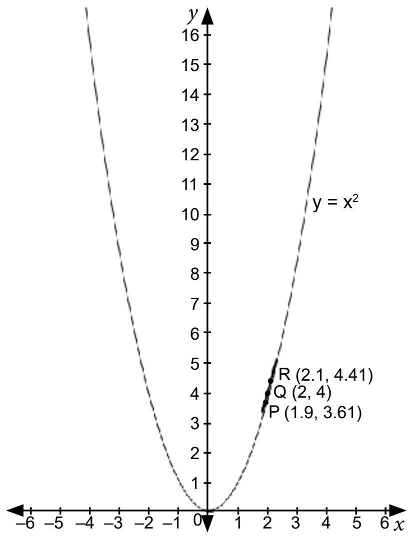
Graph of $y = x^2$ with points P (0.5, 0.25) and R (4, 16)

Draw a line through two random points P (0.5, 0.25) and R (4, 16) on the curve and calculate its slope.

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16 - 0.25}{4 - 0.5} = 4.5$$

This is a rough approximate slope of the infinitesimal tangent at point Q.

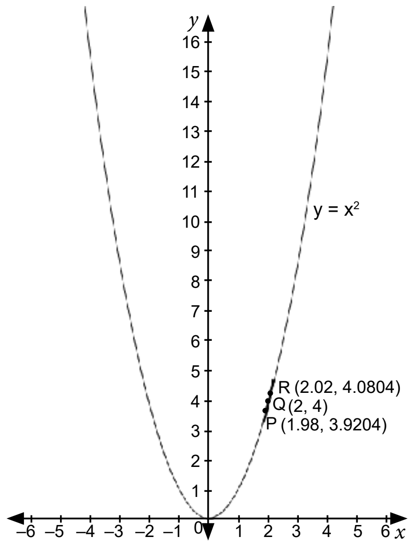
Similarly, take another set of points, say, P (1.9, 3.61) and R (2.1, 4.41), which are closer to the point Q (2, 4).



Graph of $y = x^2$ with points P (1.9, 3.61) and R (2.1, 4.41)

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.41 - 3.61}{2.1 - 1.9} = \frac{0.8}{0.2} = 4$$

Now as we move points P and R further close [i.e., P (1.98, 3.9204) and R (2.02, 4.0804)] to point Q on the curve to find a better approximate value of the slope of tangent on point Q.

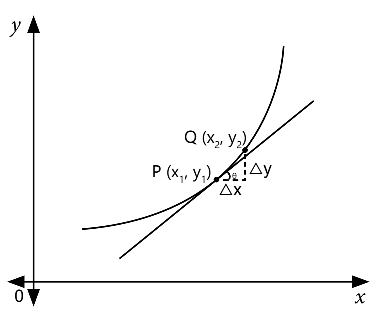


Graph of $y = x^2$ with points P (1.98, 3.9204) and R (2.02, 4.0804)

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.0804 - 3.9204}{2.02 - 1.98} = \frac{0.16}{0.04} = 4$$

As the points P and R get closer and closer to the point of interest Q, the line becomes smaller and smaller while the slope of the line changes. When the points P, Q, and R are closest possible, the line becomes a tangent and the slope of this line gives the best approximate value of the slope of the curve at the point of interest (point Q).

Another way to look at it is as follows.



Graph of slope of a curvilinear function

As seen, the slope of a line = $\frac{y_2 - y_1}{x_2 - x_1}$

For a curve, x_1 is taken very close to x_2 and y_1 is taken very close to y_2. Thus, $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ are very small.

Also, $\tan\theta = \frac{\Delta y}{\Delta x}$.

Thus, a slope of a line can be characterised using tan θ, which is the angle made by the tangent to the curve at the given point and the horizontal axis.

Why is the limit so-called?

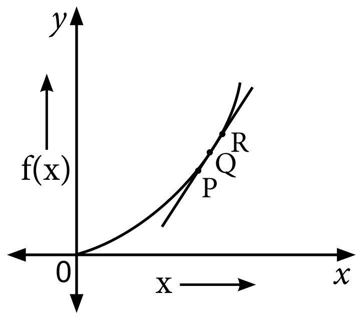
The word 'limit' is the best descriptor of the value of the function, when there is the smallest change in the value of the variable, say x. Formally, the 'limit of the function f as x goes to c is t' can also be rephrased as 'As x approaches c, the value of the function f gets arbitrarily close to t'.

In real life, when a chemical reaction between two chemicals takes place, a new compound is formed as time passes. So here, the new compound is the limit of a function as the time approaches infinity.

Similarly, tossing a coin gives a head or a tail. To know the probability of the outcome, we may flip a coin many times, making repeated trials. Here, as time approaches a more considerable period, the number of heads becomes equal to the number of tails in general. So, the limit of tossing a coin is the probability of getting an equal number of heads or tails as time approaches infinity.

Continuity solves the conceptual challenge of instantaneous values

The entire concept of limit hinges on how effective is the chosen infinitesimal value in detecting the rate of change or the slope of a function at the chosen instant. For example, while finding the best approximate value of the slope of $f(x) = x^2$ at the point Q (2, 4), the points P and R are moved as close as possible to Q. The reliability of the computed rate of change at an instant, is measured in terms of how consistent is the value of the rate, i.e., how close are the slopes of the tangent at P and tangent at R.



The slope of the function f(x) at point Q

One of the more obvious ways and means of seeking consistency is to look for the values of the rate at instants very close to the chosen instant. It is easily appreciable that the consistency of the rate of change of a function would be considered higher if at the two instants around the chosen instant – before and after – the computed values of the rate (before and after) are the same as the rate value at the chosen instant.

Thus, the infinitesimal value should be such that it can detect sharp variations in the values of the rate, closest to the point of interest (for finding an instantaneous value). The chances of capturing any sharp variation increase as the infinitesimal becomes smaller (and comes closest to the value of the instant). Any detected sharp variation declares a lack of continuity at that point.

Continuity is an important consideration for finding derivatives, it helps to know if a function may not have a derivative at an instant (non-continuous functions do not have derivatives, we can know this without having to attempt a computation of the derivative), but it is not a necessary condition for computing the non-derivative value of a function over a range.

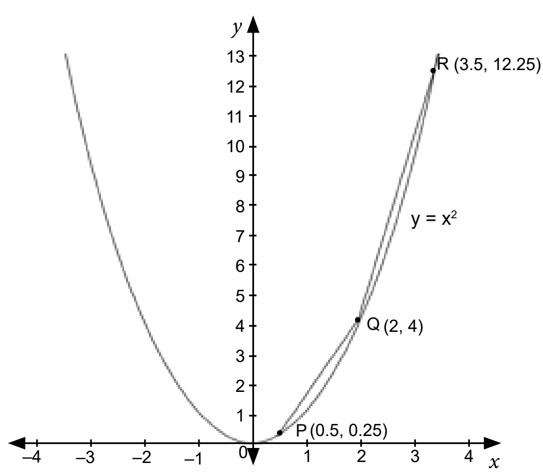
A digital recording of a song is an example of a continuous function. The digital recorder records tiny bits of sounds several times a second which may provide sufficient data for a computer to replicate the singer's overall performance while singing.

The growth of nails in human hands and feet is another example of continuous function. The nails grow at an average rate of 3.47 millimetres (mm) per month, or about a tenth of a millimetre per day. It grows and slides along the nail bed (the flat surface under the nails), giving strength to the nail. This process continues until the death of a human being. However, some factors that affect this continuous growth of function are age, location, season, hormones, health, etc.

Ascertaining continuity at a point

Let us graphically see how important is the choice of infinitesimal in appreciating the concept of continuity of a function (as evident from its curve) at a point. For a function to be continuous at a point, it is obvious that the slope of the tangents on the points just before and after the point of interest is nearly the same.

Take any random point on the curve $y = x^2$, say, P (0.5, 0.25) and R (3.5, 12.25) below and above the point of interest (point Q), and join points PQ and QR.

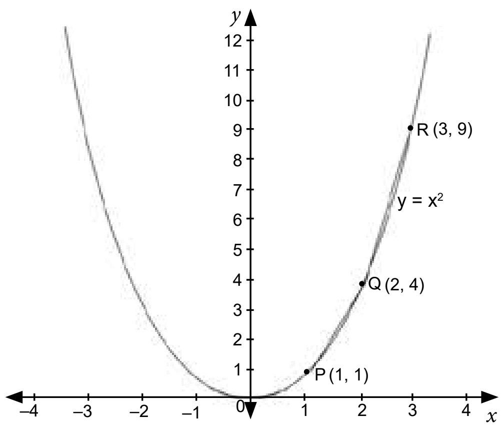


Graph of $y = x^2$ with points P (0.5, 0.25) and R (3.5, 12.25)

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0.25}{2 - 0.5} = \frac{3.75}{1.5} = 2.5$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12.25 - 4}{3.5 - 2} = \frac{8.25}{1.5} = 5.5$$

Now we move points P and R closer to point Q to calculate a better approximate value of the slope of the curve at Q. Let us choose P (1, 1) and R (3, 9), which are closer to Q (2, 4).

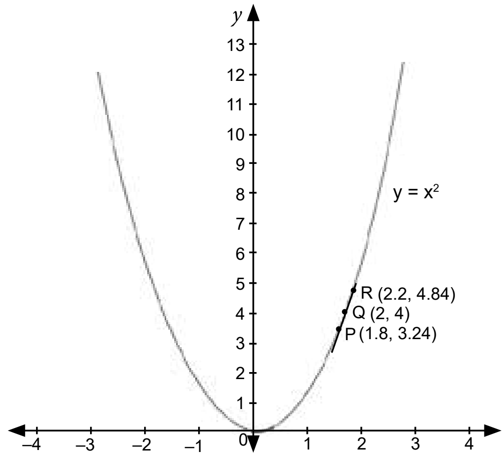


Graph of $y = x^2$ with points P (1, 1) and R (3, 9)

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 4}{3 - 2} = 5$$

Similarly, take another set of points, say, P (1.8, 3.24) and R (2.2, 4.84), which are further closer to point Q (2, 4).

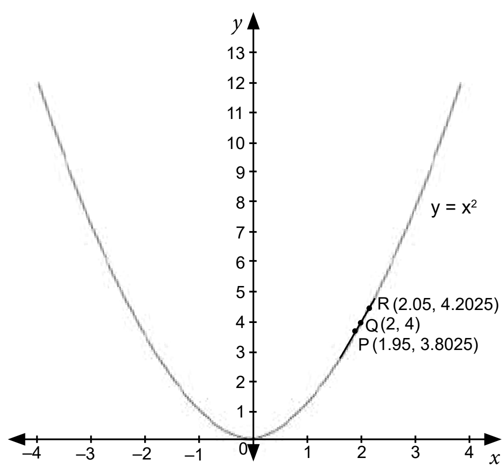


Graph of $y = x^2$ with points P (1.8, 3.24) and R (2.2, 4.84)

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3.24}{2 - 1.8} = \frac{0.76}{0.2} = 3.8$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.84 - 4}{2.2 - 2} = \frac{0.84}{0.2} = 4.2$$

Now moving very close to Q, take points P (1.95, 3.8025) and R (2.05, 4.2025).



Graph of $y = x^2$ with points P (1.95, 3.8025) and R (2.05, 4.2025)

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3.8025}{2 - 1.95} = \frac{0.1975}{0.05} = 3.95$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.2025 - 4}{2.05 - 2} = \frac{0.2025}{0.05} = 4.05$$

As points P and R get closer and closer to the chosen point, Q, the lines PQ and QR will coincide, eventually forming a tangent at point Q. The closer the points P and R are near the chosen point of interest Q, the closer are the values of the slopes at these points, suggesting a continuous function.

Other examples of continuous change beyond motion

Average or indicative rate of reaction is an important characteristic of chemical reactions. It is an essential parameter in large-scale manufacturing of chemicals, drugs, and household chemicals. For instance, knowing the rate at which products are being made and the bottlenecks (which may mostly be due to the lower-than-anticipated speed of reactions) production process can be fine-tuned.

For a chemical reaction, the change in concentration of reactants or product per unit time (such as second, minute, or hour) over a given period of time is called the average rate of reaction. And in a reaction, the rate of change of concentration of the reactants or products at a particular instant of time is the instantaneous rate of that reaction at a specific instant of time.

An interesting feature of the rate of reaction is that it continuously changes during every reaction – it depends upon the residual concentration of the reactants (which decreases with each passing instant of the reaction). A reaction never proceeds at the average rate of reaction. To really understand a chemical reaction, we need to go beyond the average rate of reaction.

Stock market intraday-trading involves traders buying and selling financial instruments based on fluctuating prices on the same day.

The trader makes a profit or loss based on the instantaneous stock price. This signifies a continuous process.

Whereas, in long-term investment, we look at the average of the stock prices and then invest in those stocks, which gives a good average return

$$\text{Average share price} = \frac{\text{Total cost of the shares purchased}}{\text{Number of share}}$$

Simple and compound interest rates are a matter of common knowledge (if not understanding) and experience that we could easily extend to broaden the appreciation of the difference between the basic idea/concept of average and instantaneous values of quantities that frequently or continuously change in time or space (i.e., change with change in position). It may also be added that when we talk about the 'real world change,' it implies that we cannot completely predict the change.

Simple interest rates are kind of 'average' of interest rates. The same flat interest rate will be applied for computing the interest amount on a principal over a period of time. The interest is assumed to be the same amount every day in that period. On the other hand, the compound interest rate resembles the idea of instantaneous interest amount – which varies by the day, week, month, or whatever period of compounding – over the deposit period.

This is to bring out that the instantaneous value of the interest amounts would behave differently under simple and compound interest situations.

This chapter is excerpted from `Calculus for Professionals,' Volume I, co-authored by Sandeep Srivastava and Dr Garima V Arora.

Notes on slope, continuous quantities and differential equations

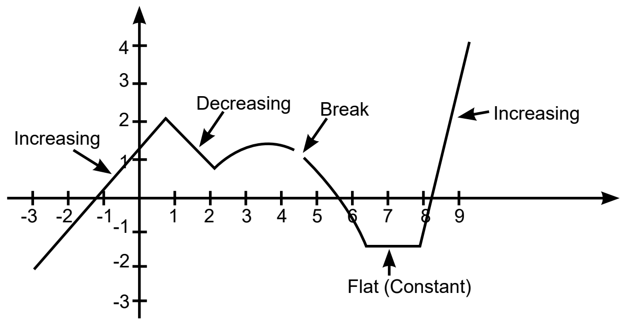
Note 1 Slope is important

The slope gives the following information about a function:

- Steepness of a function.
- Direction of change of the curved line.
- The slope describes the degree of sensitivity of the dependent variable (y) on the independent variable (x) for a function y = f(x), i.e., the quantum of change in y due to infinitesimal change in the x. For example, a slope of 4 at a point means the y-axis will grow 4 times the (small) change in the x-axis.
- The slope of the function helps us compare any set of functions to know if they are parallel, perpendicular, or converging and the rate of convergence.
- The maximum and minimum value of a function – local (within a limited range of the variables) or global (over the entire range of values).
- The slope can be used to find whether the function increases or decreases after the location of the point of maxima or the minima. Indeed, the most important characteristic of non-linear functions is their slope.

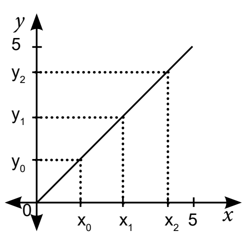
Note 2 Nature of the graphs and slopes of the functions

Calculus is constructed on functions; a familiarity with the nature of the graphs of functions is required for understanding calculus. The graphs of the functions may be increasing or decreasing. They may be flat or may not even exist. They may occur with breaks or no breaks. This information about the nature of the graph speaks about the slope of the curve at various points.

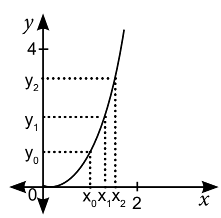


Nature of the graphs - Increasing, decreasing, break, flat

The instantaneous value of (constantly) changing quantities can only be found through the knowledge of the rate of change at all instants. The slope is the way to find the rate of change of the function, i.e., it describes how rapidly the outcome of a function changes with a unit change in its input(s) at various points in its domain. It tells about the steepness and direction of the lines and curves. Graphically, the rate of change is the slope.



Graph of uniform rate of change



Graph of non-uniform rate of change

Mathematically, the rate of change is the ratio of the change in the value of the function y, or f(x) due to a corresponding change in the value of x.

$$\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{(y_1 - y_0)}{(x_1 - x_0)}$$

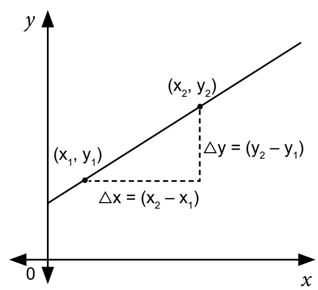
Constant slope of a linear function

It is easy to find the rate of change (or slope) of linear function, where the rate of change is constant.

Let (x_1, y_1) be the point where the slope of the line is to be determined. Let us take any random point (x_2, y_2) on the same line.

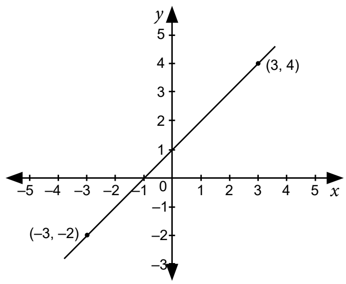
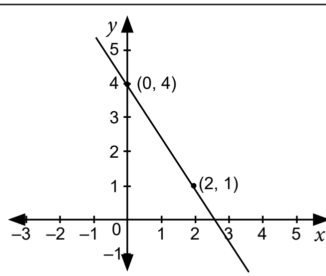
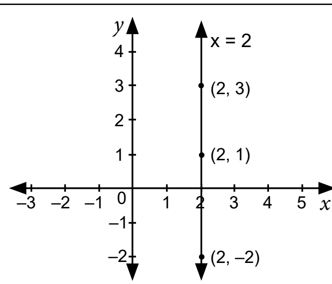
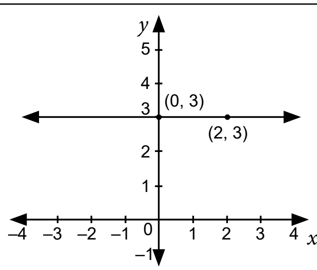
Then, the slope of the straight line is given by

$$\text{Slope, m} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Graphical representation of a straight line

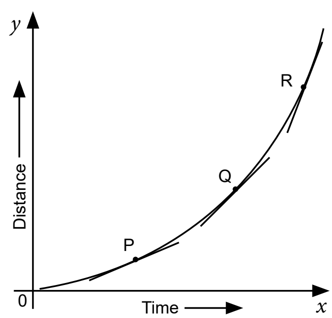
Now, let us consider the four cases mentioned below:

Case	Graph	Slope
Case I Consider a line joining the points (3, 4) and (–3, –2).		An increasing function describes a positive slope. $m = \frac{4-(-2)}{3-(-3)} = 1$
Case II Consider a line joining the points (0, 4) and (2, 1).		A decreasing function describes a negative slope. $m = \frac{1-4}{2-0} = \frac{-3}{2}$
Case III Consider a vertical line x = 2.		A vertical line describes an undefined slope. $m = \frac{3-1}{2-2} = \frac{2}{0} = \infty$
Case IV Consider a line joining the points (0, 3) and (2, 3).		A horizontal line describes zero slope. $m = \frac{3-3}{2-0} = 0$

From above inferences, we can say that the slope (or rate of change) of a linear function is always constant.

Varying slopes of a curvilinear function

The slope (or the rate of change) of the curvilinear function is not constant and keeps changing along the points on the curve. A tangent at a point on a curve is a straight line that best approximates the slope of the curve near that point.



Tangents at various points on the curve

Tangent best approximates the slope of curve

Let us see for ourselves how the slope of a curve at a point is best approximated by the tangent at that point (there can be only one tangent at a point on a curve).

We start with an ellipse with tangent PR at the point Q. From the image (i), it is not evident that the slope of PR is the same as the slope of the ellipse at point Q.

On enlarging the image and reducing the tangent PR (image ii) and we still cannot see the relationship between the slopes of PR and the ellipse at Q.

We further reduce the size of the tangent, and enlarge the diagram to view the relationship between the tangent and the curve of the ellipse.

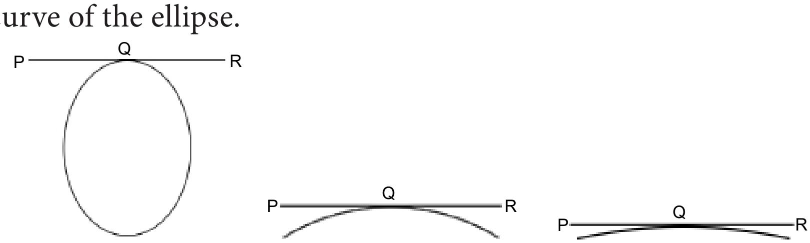
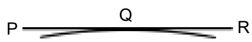
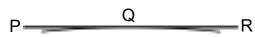


Image (i) *Image (ii)* *Image (iii)*

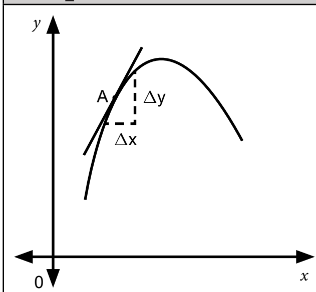
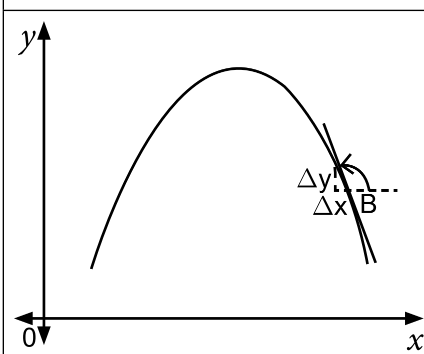
Interestingly, as we keep enlarging the image and reducing the size of the tangent to know the preciseness of the slope of the curve at point Q, we will notice that a very small area of the curve (or a point) coincides with the tangent.

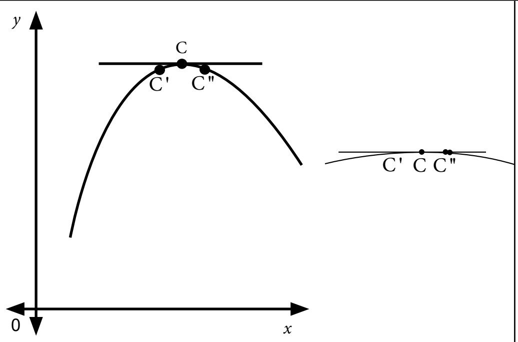




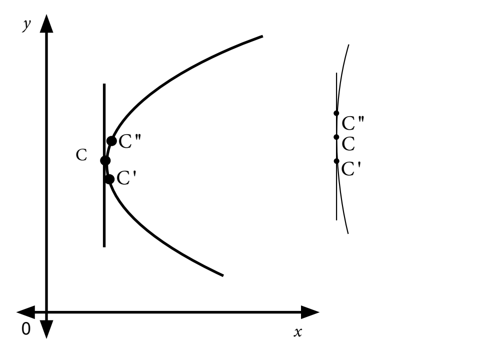
As we have already discussed, it is easy to calculate the slope of a straight line. To find the slope of a curve at a specific point, we find the slope of the tangent line at that specific point, it provides the best approximation. In the diagram, the slope of the tangent PR is the best approximate value of the slope of the elliptical curve at Q, as the tangent PR coincides with the curve at Q.

Characterisations of slope for a curve

Graph	Slope
	• The tangent to the curve at the point A is tending upwards when moving from left to right, which is a property of increasing functions – positive slope. • The tangent at A makes an acute angle with the horizontal.
	• The tangent to the curve at the point B is tending downwards when moving from left to right, which is a property of decreasing functions – Negative slope. • The tangent at B makes an obtuse angle with the horizontal.

<table>
<tr>
<td>
</td>
<td>

- As we move from C' to C and C to C" with an increase in x, the value of the function remains the same.
- The rate of change in both cases being 0. Thus, a horizontal tangent has slope 0.

</td>
</tr>
<tr>
<td>
</td>
<td>

- As we move from C' to C and C to C", the value of the function increases for the same value of x.
- The slope is not defined for a vertical tangent on the curve at a point.

</td>
</tr>
</table>

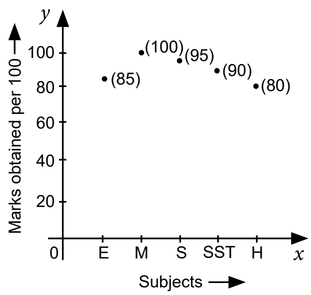
Note 3 The nature of input and output

Nature of input

The nature of input effects the nature of output. Let us explore how input values for a function may be different.

There is a world of things and situations that are quantified through the act of counting, and a world that is quantified through the act of measurement. The marks obtained by a student in an exam is quantified by the evaluator by first counting the marks obtained in individual questions and then adding them all up, whereas the weight of students in a class is quantified by measuring the weight of individual student on a weighing scale. Recall, functions take some kind of quantities as inputs and produce some other, or similar kind of quantities as output, which could be obtained by counting, or by measurement.

The nature of quantities obtained by counting is what we call as discrete. For example, marks obtained by a grade X student in all subjects in a school have discrete values, graphically seen as follows.



Graphical representation of discrete quantities

Discrete does not necessarily mean integer value, it only means that the possible values are definite and known; for example, the marks in a subject could well be 85.5, 85.25 or 85.75, but hardly 85.55.

The nature of quantities obtained by measurement/computation is typically what we call continuous, expressed using real numbers. Continuous quantities can take any value in an interval. For example, the aggregate percentage value is technically discrete because aggregate percentages are calculated by dividing the marks obtained by the total marks, which itself are discrete. However, percentage data is often treated as continuous for the reason that it can take any value from 0 to 100%.

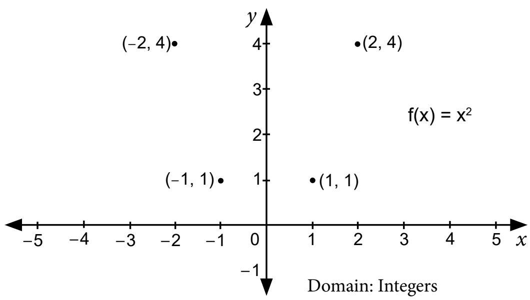
Thus, input values could be discrete or continuous, depending on how they originate – out of counting, measurement, or arithmetical computing.

Discrete or continuous function – The nature of output

The nature of output could also be discrete or continuous. The effect of the function on the nature of outcome is obvious; for example, $f(x) = x^2$ accepts inputs as negative and positive numbers but the outputs are only positive.

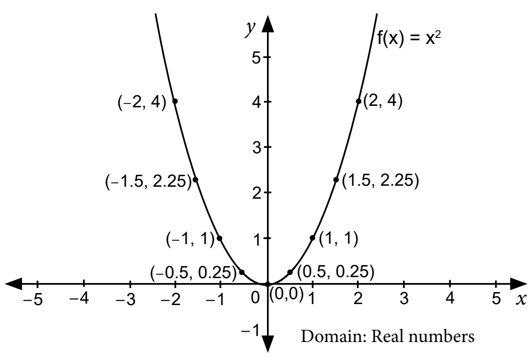
Let us take a function $f(x) = x^2$ and take different values to know how this function behaves.

When input values are taken as discrete, say, −2, −1, 0, 1, ..., the nature of output is also discrete and it is 4, 1, 0, 1, ..., corresponding to the input values in the given function.



Graphical representation of function for discrete input values

On the other hand, when the domain is an interval (say [−2, 2]), the output or the range is also an interval (here [0, 4]) and takes every value between −2 and 2, and hence continuous.



Graphical representation of function for continuous input values

Thus, for a different domain of input values, the same function $f(x) = x^2$ will have a different range of the output values.

The nature of output depends on the nature of input and the nature of the function (i.e., the kind of 'processing').

The idea of a function being continuous is especially important in calculus.

Note 4 Mathematical expressions using derivative (Differential equations)

Recall that algebraic expressions are combinations of constants and variables which are put together using mathematical symbols and algebraic equations are expressions that are set equal to zero. It is indeed fascinating to consider that equations can incorporate changing conditions and the rate of change through the use of derivatives and rates of change. Such equations are common, they are mathematical tools used for modelling and analysing everyday situations and scientific conditions that involve change and relationships between variables. These 'derivative equations' are mathematical termed as 'differential equations'.

Do not be startled if we say that the idea and application of 'differential equations' is a primitive biological, animal instinct and ability. We are adept at using differential equations intuitively.

For starters, in a race, gauging and reacting in the heat of the moment to the increasing or decreasing distances between the runners as things change dynamically involves using the brain's raw/god-gifted logical abilities and spontaneous calculations. What we do intuitively, formal mathematical modelling of the situation would involve calculus! The conscious and engaging assessment and extra push mid-air when jumping across a ditch ensures that the jump is successful. Again, a mathematical modelling of decision making in the situation would be based on the derivative (rate of jump) and anti-derivative (the extent of jump).

Wherever there are changing quantities in the 'equation' of thing, the situation is mathematically expressed as differential equations. These equations can be used to configure everyday life to rocket science. The laws of nature and dynamism in science and maths can easily be explained through differential equations. Think of the way we go across a busy road – constantly juggling with the estimated speed of the vehicle (rate of approach), the closing distance between the fast-approaching vehicles and the person crossing the road, the distance left to cross the road, speed of the person crossing the road, as well as the obstructions on the way (the other people crossing the road from the opposite direction, for instance); it is a fairly complex situation of changing dimensions, but most of us have gotten it right every time.

Some examples of differential equations in real life:

- Any change in human body temperature is a response to changing conditions outside and within the body, such as ambient temperature, the type of food eaten, the type of clothes worn, the type of activity performed at that particular time (for example, exercising would increase the heart pumping and blood circulation rate, thereby increasing the temperature), and more. So, 'in the equation', the temperature of a human body responds to various changing conditions underlying it.
- For calculating the time required to drain a tank full of fluid, differential equation comes into play. Draining time depends on various factors like the volume of fluid, the air pressure, the height of the tank, the density of the fluid, the rate of flow, the size of a draining hole, and more. Any change in the above factors may impact the time taken to drain the tank. For example, if water and petroleum are put in two similar tanks, the time taken to drain them would differ due to the difference in their density. Similarly, if the size of the draining hole is small, it would impact the time taken to drain the fluid. 'In equation terms', factors like the size of the hole and height of a tank are constant terms for a specific tank for all fluids, while density, the volume of fluid, and air pressure are differentiated with time to estimate the time taken to drain the fluid.
- The value of the National income of a country is dependent on various factors like general price level, aggregate demand, aggregate supply, compensation to employees, saving rate, government policy, and more. These factors depend on the inflation rate, total production of goods and services, wage rate, marginal propensity to consume, etc. All the factors are

interlinked and dynamic in nature. For example, the general price level of a country increases due to an increase in the inflation rate, which may change due to government policy or a change in supply. 'In equation term', compensation of employees and saving rate are constant terms for a period while general price level, aggregate demand, and supply are differentiated with time to know the value of national income.

- The differential equation is used in a video game to determine the rate of motion of an object. For example, consider the static force diagram for a ball rolling down a ramp. Knowing the time duration in which it will roll down depends on various factors like gravity vector, mass of the ball, and the angle of the ramp (its normal vector).These factors are further dependent upon the net force applied on the ball and the acceleration of the ball in that frame. So, 'in equation terms', the mass of the ball and gravity vector is considered constant, while other factors are variable and differentiated with respect to time.

An interesting aspect of differential equations is that unlike algebraic equations and the math we know, the 'solution' of differential equations is not a quantity (or a set of quantities) but another function. Such a solution might be expected because when we deal with derivatives, we essentially deal with functions. It means that differential equations give us a 'modelled' behaviour of things, not a particular instance of behaviour. And there is often a set of solutions for a given differential equation.

A few famous equations in physics which depict the rate of change are:

Force = Mass × Rate of change of velocity
Power = Voltage × Rate of change of charge
Momentum = Mass × Rate of change of distance
The conceptual exploration of derivative (and the related idea of 'anti-derivative and derivative equation') concludes here.

It's Your Convenience World, finally

This is for real, for once

> *"Welcome to the New World Order – the IYCWorld (it's-your-convenience-world). The power to choose yourself and engage in your chosen socio-economic proclivities, around the world, at the click of a mouse is a force that will transform your life like nothing before. And your choice will be your limit in the skyless e-universe. IYCWorld is only as good as you demand it to be ... try as they might, the local socio-economic dimensions cannot stop your will and convenience to rule."*
>
> – *Sandeep Srivastava, 2001*

This is an excerpt from the 2001 book 'Embracing the Net,' published by FT.com (Pearson, UK,) co-authored by Soumitra Dutta, currently Dean Said Business School, University of Oxford, and Sandeep Srivastava. The extract was the stated overarching vision and strategic direction for the digital economy in the Third Industrial Revolution (3IR), post the 2000 dot-com bust.

Twenty-three years later, and a decade in the Fourth Industrial Revolution, 4IR, the vision and strategic prescription is just as valid and robust, literally nothing even to be tweaked. Indeed, we have come a long way in the right direction, the emergence of the 4IR as the hard-infrastructure for Society 5.0 is just the needed 'physical enabler'. However, we are far too away from transformations on the ground.

It is not too hard to locate why we are still as long a way from Society 5.0, in Japan as much as in every nation of the world – the complementing soft-infrastructure of the 4IR is missing in action. A vast majority of the educated adults of our times cannot harness 4IR, and it is turning for the worse. Education – the technology of raising best-potential adults out of every child – has turned out to be the intractable complication for all of humankind.

However, mathematisation of mathematics education is a masterstroke for ushering in an educational renaissance. For, learning mathematics is peerlessly personalisable and most objectively evaluable. Besides, mathematics is the easiest domain of knowledge to learn; every one of us is born with all the mathematical logic that is there is to be discovered, waiting to decipher the order in the nature. Success in mathematics education is the only first step way to kick start the larger educational reformation, and we can go on.

Above all, 'Cent Percent Mathematics' is no more than 50 hours of conversation for the entire K-12 curricula. And language is no barrier. Cent Percent Mathematics is all in public domain, the evidence of which are the two case studies. Mathematics-led educational revolution is real now, ready to more than complement 4IR and set off a virtuous cycle of growing economic dignity to every one of us.

Be ready to play your part, in mathematising your own mathematics, and experience economic miracles for self, family, and community.

It is essential to realize that science does not offer a complete knowledge of the mind, although we do experience its mystery and enormous energy. However, it is clear that energy is a vital prerequisite for performing mechanical work.

The mental processes of the mind are essential for performing creative, innovative work, the difference lies in how people utilize the power of their mind. Everyone knows the saying that "an empty mind is Devil's workshop," so without a meaningful purpose, people might spend their mental energies on destructive work. On the other hand, mental energy can be utilized for creative or innovative work as well as improving quality of life.

A mathematised mind is highly predisposed towards seeking and seeing order in all things around.

– Ramjee Prasad, 2012