# Mathematising MATHEMATICS

MATHEMATISED THINKING INFINITES 4IR

CATALYSING GRACIOUS SOCIETIES

RAMJEE PRASAD
SANDEEP SRIVASTAVA

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ABRIDGED VERSION

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# *Mathematising* Mathematics



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*For my grandchildren –*
*Sneha, Ruchika, Akash, Arya, Ayush and Shreya –*
*Who inspire me to look ahead with joy and zest.*

*Ramjee Prasad*

*To the truly educated – reader, gracious, assertive, humanist,*
*and proud citizen of the society.*

*Sandeep Srivastava*

Dear Readers,

You must know that this book is in a genre of its own – an academically rigorous book for us all globally. For far too long, academic books have targeted only a fraction of the population; even scholastic achievement is taken to be the preserve of the gifted ones.

The tone, organisation of content, conceptual intensity, and expanse of the book qualify it to be an academic work. However, the language is akin to literary writing, and the presentation is 'textbookish' to facilitate an inalienable grip over the flow and substance.

Yet, in the jest to enliven academic concepts, their taut boundaries may have been infiltrated.

Expectedly, the book would not be a familiar feel for all readers; it straddles across the unbridged academic and 'trade' (general public) genres.

We wish hope this inventive format of the book is appreciated.

“Openness of mind
is a gateway
toward creating
a **global** village”


– Ramjee Prasad
Unlock Your Personalization
Aalborg University Press (March 1, 2012)

This book is also one sequel to 'Unlock Your Personalization.' One end of this book is 'Mathematising (our) thinking' to root mathematics as a language of all social institutions and processes.

'Unlock Your Personalization' promotes an innovative and novel approach to achieving a good quality of life.

Life is short, and its limits are apparent. Living should bring happiness and pleasure, but most people have to cope with enormous problems stemming from heavy workloads, stress, and anxiety. In our post-modern, techno-science world, every effort is being made to achieve a high standard of living. Still, few people find an effective solution for relieving stress and achieving their objectives in life.

– Ramjee Prasad

# Content

**This Abridged Version has Section I and the Annexure.**

# Preface

## *Thinking Humankind, All*

Swami Vivekananda prophetically called upon us to 'make men first'. He reminded us of the on-the-ground truth that even if governments give us all we must have, *'where are the men who are able to keep up the things demanded.'* Unsurprisingly, the italicised phrase encapsulates the most crippling crisis before humankind a century after its articulation by him. The phrase quite sits at the heart of what the book is set to catalyse – (human) development revolution – by galvanising us all to be spirited humans in the ever-intensifying Sci-Tech (Science and Technology) and AI (Artificial Intelligence) era.

More simply stated, this book is about you, your family, and the community you are nested in. It is most peerlessly so along multiple dimensions. The book is about contemporising you – your knowledge and skills – to the technology of the times – Artificial Intelligence (AI). The book will seed in you the exemplary life to thrive in the fast-realising world of inorganic intelligence, an unprecedentedly malleable future. The book will lay bare the designs and destinations on the Fourth Industrial Revolution (4IR), or Industry 4.0 highway.

The book also decodes why 4IR is struggling to hold its ground, let alone accomplish its promise, and how the context of your family and community cannot be future-proofed if 4IR fails in revolutionising social infrastructure – education and health for all 8 billion of us, and the (economic) dignity of all adults.

The only business of societies and humankind is same-outcome education of all children@18. Health is on the cusp of education and economics. 4IR is our last opportunity to trigger and sustain an unexceptionally global socio-economic renaissance.

**Educated blindness – A world hiding in plain sight**

Inattentional blindness is a biological limitation, our 'brain's fault'; we cannot really blame ourselves. It is personal in nature, and there is not even a remote chance that two random people could have overlapping inattention to similar things over different times and spaces. On the whole, it all does even out, and no loss or hurt visits anyone; for example, things that get the attention of the sexes are complementary to some extent, and together, a couple covers up for one another (one may miss blue tea roses, the other violet tea roses in a walk through a garden). The worst manifestation of inattention is mostly embarrassment.

There is a social correspond of inattentional blindness, a creation of our socio-cultural conditioning, our 'education's fault'. Unsurprisingly, it is best addressed as 'educated blindness', most acutely and almost universally noticeable among those formally educated, and the longer the formal education, the more likely is the 'affliction' with educated blindness. The only apparent similarity 'educated blindness' bears with its biological twin is that we cannot really blame the educated individuals; the cultural as well as the formal education system is riddled with holes.

Educated blindness is visible and veritable conditioning of thinking, learning, observation, acting to preserve (more significant, long term) self-interests and assertion of moral being. Its worst manifestation is thin-cultured adults, globally with an iron-curtained worldview, a rather narrow worldview, and increasingly transmitted and transplanted worldwide with the help of the most uniform social institution across the world, the K-12 formal education system.

More specifically, educated blindness is reflected in the substantive majority and the 'toppers' of the formal education system falling into a new 'average' – individuals out to pursue similar kinds of careers, having the same meaning of success in career (and not seeking professional stature), same meaning of being rich, the same way of becoming rich, marginalised larger socio-cultural identity, a culture unto themselves and so on. It is the reason behind indifference to unconscionable inequity in wealth and income, fractured society, treating climate mitigation as a new gold rush rather than a social and humanitarian challenge, 'professionalisation' of the social sector – education, health and civil society, and many things else.

**Educated citizenship – The vital soft-infrastructure**

History is sharp about each new economic revolution stepping up the demands on humans. The era of the driverless cars on the street and the generative AI of the next decade would still need (appropriately) educated humanity. For instance, people 'educated to be doing something joyous' while being driven around, people educated to reinvent their businesses around driverless vehicles, people educated to feel productive and wanted in new ways, people educated to add value to the outputs of generative AI, and more would be in demand. We have to recontextualise what it means to be educated.

However, it seems making humans (raising adults out of infants) is now a long-lost art for humanity. The more 'developed and advanced' the nation, the more vigorous and devoted the institutionalisation of this 'phase of raising, educating' children. Education is the name of all that happens to make a 'dignified, cultured adult' out of infants. Human infants do not grow into humans on biological DNA (all other animal infants do); it takes (more than) a village to raise every child over two decades. Education is 100% social – among role-model adults, peers (ideally not same-age), and the 'real world' (nature and community).

The first, primary goal of nurturing educated adults is to ensure their economic dignity, without exception. The secondary, but not lesser, goal is thickly/deeply cultured adults who live and enrich their chosen socio-cultural contexts, such as active civic constitutionalism.

**4IR – The omnipotent hard-infrastructure**

To avoid getting the wrong end of the stick, let us begin by emphasising that 4IR represents an entire array of digitised technologies, irrespective of how it is formally defined across institutions and experts. The digital foundation of 4IR (including Big Data), together with AI, is materialising a sea of appropriate technologies to best empower our individual choices. Many of these technological imperatives would require the frontier of science to be expanded like never before and also multiply the intensity of the mathematisation of science (and social sciences, and everything else too). Fortunately, research and innovation processes and resources are transforming to seed unprecedented development of science and technology. 4IR is a dream infrastructure for a socio-economic revolution for all, a first for humankind.

**Society 5.0 – Social outcome of 4IR**

Society 5.0 is defined as a human-centred society that balances economic advancement with the resolution of social problems by a system that seamlessly integrates cyberspace and physical space. The term "Society 5.0" was introduced by the Japanese government as part of their "Fifth Science and Technology Basic Plan" in 2016 to refer to society that evolves with 4IR. The plan presents the hunting society (Society 1.0), agricultural society (Society 2.0), industrial society (Society 3.0), and information age society, 4IR (Society 4.0). Much like the way 4IR represented the entire technoscape of the times ahead, the gamut of the socio-cultural world associated with 4IR is represented by Society 5.0.

A bit of economic history should help us understand the genesis of Society 5.0 and how a more organic collocation of social and economic development is non-negotiable. The Iron Age triggered a sense of private ownership of anything (land, forests in those times); iron swords, for example, could cut trees to create fields, and animal-drawn carts with iron-rimmed wheels could go way beyond the communal land to claim private ownership. Later, iron weapons made larger-scale war and killing possible, and the sense of private property extended to taking away what belonged to others. Unlike the weapons in the Iron Age, Bronze Age weapons were heavier and did not have strength or sharpness or iron weapons. As pertinently, the driver of 'privatising properties' was the appropriation of extra (societal) resources and later deployment of the differential resources for gaining property.

The race to private property has not abetted even after 3000 years. Worst, the ownership of immeasurable tracts of land (and building as a proxy for space similar to land) is the hallmark of being rich.

Almost as a default implication of private property, economic development, and the good of the entire society have never reconciled to date.

Economic development has come to mean a varying degree, exploitation of communal/societal resources and trust for 'privatising the same', to benefit a few at the cost of the entire society (to which the few may belong). Resource differential still rules the roost, and for all the time in history, it has almost always been the monetary capital. Interestingly, around the cusp of 3IR and 4IR, broadly between the mid-1990s and early 2010s, knowledge was expected to be an equal resource differential, but it is almost back to the historical reality of monetary capital again.

Another (very) long story short, it is rightful to hope that in the 4IR Age, the critical resource differential shall be individuals – their ingenuity, industry, and integrity. In other words, the education system is in its broadest (true) sense. The entire socio-cultural and economic ecosystem is now almost equally accessible to all through the Personal Multimedia Communication revolution. In effect, let us hope and work towards the differential being what individuals can imagine of 4IR and value creation through it for self, family, community, as well as humanity (remember, it is a global village now.)

Interestingly, the website of the Office of the Cabinet, Government of Japan, mentions that Japan aims to be the first state to achieve a human-centred national society, named Society 5.0 (following a particular hierarchy of the evolution of humankind). Such a society could be visualised as one that will create and nurture a socio-economic environment that best facilitates one and all to enjoy a high quality of life, exemplified by proactive and productive citizens, every one of them. It elaborates that innovative culture, diverse technologies, and the social integration of the two into the fabric of the nation are how Society 5.0 will crystallise.

Indeed, over to 4IR and popular imagination.

# Section I

## AI and 4IR – An inexplicable crossroads for humankind

'Unite to Regulate (Generative) AI' is the new battle hymn of the global political rhetoric, unseating climate change in just a matter of months. 4IR is defying its DNA even after a decade in action – it is turning (net) producers to (net) consumers, crushing economic dignity to dust for an ever-increasing fraction of humankind.

AI and 4IR are inherently about comprehensively personalising the world for every one of us, to empower every one of us to be best educated, healthy, and live with dignity in a truly democratic society (and nation). The two embody the grandest designs for an assuredly happy earth.

Instead, we are staring at an unimaginable and exceptionally collective future – adults, families, communities, and societies disabled from powering their survival and growth. The most gratifying thread throughout history is that families and communities made ends meet, war or peace. All (able) adults were net producers and paid some form of taxes to the state. It is unnatural to humanity to even think of something like universal basic income to a fast-growing fraction of net consumer adults.

Of course, the book is all about negating this possibility. Mathematised thinking is humanity's gravest miss. We present a peerless action plan to mathematise our thinking. To enable every one of us to plug into the truly global and omnipotent AI soft infrastructure and the next-generation 4IR infrastructure. Satya Nadella emphasises the need for a billion developers, considering it a democratising tool to facilitate easier access to new technology and knowledge, simplifying the learning curve. Lest 'a billion developers' is misread to mean 'a billion software/AI professionals,' we wish to assert that it is best read as 'a billion thinking professionals.'

A peerless human revolution is ahead of us!

Fortuitously, the mathematical rooting of this revolution guarantees its infallibility. Mathematics may just be the most explicit self-organising consciousness in us. It is anchored in the ways and working of the world around us as it gathers mass and momentum almost autonomously, on its own devices. The mathematisation of humankind is unstoppable in its reach and catalytic drivers.

# Humankind is Mathematising

*Soaring discovery, invention of Order*

The place of mathematics in society has never been in question. The pedestal accorded to mathematicians across societies volumes about it. Mathematical knowledge and skill are considered such a suprahuman occurrence that even a poor academic record in mathematics is some kind of a badge of quality of being 'normal', being creative in some ways. People feel cringed at not being good at drawing, singing, or dancing, but being poor at mathematics is a matter to bond over. Apparently, mathematical handicaps seem to bear little correlation with professional success. To top it all, mathematics skills need not be personally mastered because mathematical computations enjoy the benefit of high integrity. Misdemeanours and even frauds never go undetected for long, especially after due forensic examination.

However, all this leniency with lack of success in mathematics education started to pinch as the Third Industrial Revolution, 3IR, peaked towards the last decade of the twentieth century. Chorus for improving math education globally has been getting shriller ever since. 3IR matured with analogue electronics turning to digital electronics; semiconductors, personal computers, internet, mobile phones, robotics, 3D printing, and the like were the typical innovations of this era. 3IR brought down barriers to opportunities and resources to ignite knowledge-led, small-business entrepreneurial ventures to compete with and complement corporates.

The Fourth Industrial Revolution, 4IR, has progressed the techno-economic cart to bring more disruptive, global, and personal force to innovations. 4IR emerging technologies, such as IoT, 5G, Big Data, Self-organising supply chains, Next-generation (DNA) Sequencing, Artificial Intelligence, and Blockchain, are significantly differentiable from the Third Industrial Revolution, 3IR and the Second Industrial Revolution, 2IR. Technologies for being duly more general and original. The more general a technology, the more flourish they bring to technical progress and economic growth. The more original a technology, the more refined it is; using more diverse domains of knowledge in technology is one way to make it more original. Originality and generality trigger newer products/services and newer markets/people, respectively; for instance, generative AI has caught the imagination of employers for surgical excision of their customer service teams, just as some rudimentary AI application is already reaching all 3 billion Internet users.

It must be admitted that these technologies are similar to their 3IR roots. However, their extra thrust lies elsewhere – more science (new and refined) and more rapid innovations (shorter 'Technology Cycle Time'). Technology Cycle Time is broadly the period in which technology lives in its original suit. It involves the invention of technology, its justification as superior, purchase and deployment, sustenance/maintenance, and till the emergence of a better technology. In a way, 4IR is a 'general purpose (sci-tech) ecosystem', so vastly expansive that it will underpin infinite 'plug and play' choices for people.

**Soft is finally the hard, new power**

The feminisation of humanity is undeniably set. Multiple gender recognition is steadily mainstreaming. Corporates are softening institutional culture to retain one and all, despite customers picking up the cost of the slack (is there ever a free lunch). The notion of soft power still has its believers in the conduct of international relations, especially made possible in the times – of collective climate response, the Internet, global supply chains, migrations, etc. In many sections of societies across nations, soft is becoming quite aspirational, being equated with being wise.

The soft power in the digital age is on another plane. 3IR was hard powered, and 4IR is soft powered – driven by mostly soft – software codes, digitalised data, mathematical models, self-correcting codes, cloud storage, non-resident stored procedure codes, soft reboots, soft memory boxes, and more, and new softer dimensions.

Is there a softer of the soft? Yes, mathematical models! Besides, mathematical objects (numbers to proofs) are the steel of every example of the aforementioned soft dimensions.

Indeed, as 4IR becomes an integral part of all of us, it is no exaggeration that humankind is mathematising.

The shreds of evidence of steadily mathematising humankind are overwhelming.

**Technology – The embodied mathematics**

Technology is the outcome of applying established scientific knowledge at that point in time. Technology could be valuable as tangible products or intangible services (such as software codes). But tangible (such as a water tap or a pen) or otherwise (such as Internet search engines), all technologies are mathematically controlled and determined. For instance, every kind of pen has a predetermined flow of ink, an acceptable range of pressure on the writing tip, chosen measures of grip/holding area, and more; any sudden change in these mathematically determined parameters would make the pen unsuitable. The deeply mathematical colour of Google's search engine is a good read.

The Google search engine needs no introduction; it may be the most popular technology we use. It is rather common knowledge that the engine uses a simple arithmetical count, called PageRank, to develop a 'relative order' of the importance of webpages of millions of websites. For the curious, the PageRank algorithm does not 'see' the contents of web pages for ranking, it uses the popularity of the webpages – the link structure, how frequently the webpages are called, or hyperlinked by other webpages.

Mathematically, PageRank ends up assigning webpages a score between 0 and 1 (a probability). Hence, a PageRank of 0.5 would imply that there is a 50% chance that a click on a link (for certain keywords) would end up on that webpage. There is more – the computation of the PageRank score is not a simple probability computation, and it keeps evolving (making us all constantly guessing the exact algorithm.)

PageRank is a particular patented process of (web) page ranking, and Google also uses other ways of ranking pages, such as the mathematical model called Markov chain (a model is simply a quantitative relationship of varying things, such as the formula for the circumference of a circle in terms of its radius). Markov chain is also based on probability; it offers what is more like a sequence of possible situations, apparently random, that a given situation may transition into, with probability of occurrence of each of the situations. It is widely used for predicting changes, especially in those situations that change with time.

Its value is its simplicity – it predicts the next situation or step based only on the current situation, not how it was reached. Markov chain so mathematises the relationship of the change that it is suitable for making decisions in real-time, such as developing options of routes to take in a traffic jam, or communication transmission routes in a network. The mathematical model is such that the quality of the predictions, based just on the current state of a situation, is as good as if the predictions were based on the knowledge of 'all the past states' of the situation.

Public key encryption is the mathematics behind Blockchain. One of the most popular blockchain network applications – the cryptocurrencies – are so mathematically founded that they may be thought of as 'mathematical money', their creation and value sustenance is a mathematician's delight. It involves very complex mathematics and computational methods. Expectedly, it has developed mathematical models that have applications beyond the cryptocurrencies; the models used for pricing cryptocurrencies are so expansive and exhaustive that they could be used to estimate the 'market (or reasonable) value (price)' of many things.

The basic knowledge of the association of calculus to everything that is not straight and standard geometry prompts that 3D printing technology must be drenched in mathematical soundness. The dynamical properties of objects to be 3D printed, such as the strength of the initial conditions/steps/formations, shapes, stability of the whole print, and the sharpness of the contours, are just some dimensions that must stand the scrutiny of mathematical modelling. Differential equations are extensively used in the software used in 3D printing.

There can be no discussion on technology without taking a peek into how TikTok redefined timelines to reach the top of social applications. Eleanor Cummins, a freelance science journalist and an adjunct professor at New York University, brands TikTok's algorithm as 'all-knowing', and goes on to summarily contrast with Facebook. To her, '*whereas Facebook asks you to set up a profile, and hand over a treasure trove of personal information in the process, TikTok simply notices—or seems to.*'

TikTok may be the best showcase of how Linear Algebra, a relatively simpler mathematics, is rightly recognised as the mind of machines (help 'machines learning'.) In this context, Eleanor is so right in saying that mathematics has '*flattened humanity into a series of codes,*' it is the invisible hand at work acting more like a supernatural force moving and shaking all things around us.

New mathematical modelling is redefining medical diagnostics. For instance, the design of MRI machines is such that it is problematic for claustrophobic patients, and it is noisy for all.

Speeding up the scanning process has been at the top of the MRI research agenda. Emerging advances in mathematics came to rescue patients, doctors, and biomedical researchers. Mathematics made high-fidelity compressed sensing a reality by compressing patterns of '0' in the captured digital images, a technique commonly used in reducing the 'size' of MP3 and JPEG files. MRI scans that once took 5 minutes can now be carried out in 30 seconds using data compression models.

*Mathematics is more directly touching our lives by the day.*

**Science – The divided house**

(Natural) Science is the body of knowledge that codifies the (universal) secrets of the living and non-living nature as it operates in, on, and around earth. It is evenly split between the mathematised science (physics and most of chemistry) and the descriptive science (a good part of biology). It must be added that all biological and medical technology is mathematised, by definition; drugs, blood tests, scans, etc. are deterministic when matching with clinical diagnosis.

Mathematics is already the only language of physics, and there is a thriving community of mathematical physicists focused on the mathematical foundations of theoretical physics. Mathematical concepts and objects are the means of thought experimenting in theoretical physics. Any discovery and development in physics are automatically mathematised, whether theoretical or experimental; viable experimental setups are the work of technology, mathematics. A leading area of research in physics, quantum mechanics, is already such that a rigorous description of quantum mechanics is purely mathematical.

The story of theoretical chemistry is similar; mathematics is the means of its thought experiments. Also, it borrows from physics, mathematics, biology, and computing to further investigate and simulate molecular behaviour, develop new molecules, and develop a new theory.

Biology is following the other sciences in using the language of mathematics. It is not as mathematised because we still have to know far more about our commonness and uniqueness. This is because biology is the physics and chemistry of living beings (the latter two are essentially about the non-living world). Yes, 'that life in living' is Godly, and who knows if we will get there as humans or through AI. The secular trend is loud and clear – biology is getting mathematised; it is slow but steady.

Interestingly, the cognitive mathematisation of science started with Galileo, who married mathematics and astronomy (physics). It would help to know that what we today call science was known as natural philosophy in his times until the early nineteenth century. His views on mathematics as the language of science remain the last words on it to date. He metaphorically referred to the ways of nature as the 'book of nature' that can only be read through mathematics.

To him, mathematics was the language of nature; he saw a mathematised nature. He also added that a philosopher has to be a mathematician, and in the process, he separated 'pure mathematics' from mathematics that helped in understanding the real, physical world. In the early seventeenth century, he knew that the mathematical view of the workings of nature was too complex to be readily appreciated by people. This view also acknowledges that mathematics was too idealised in a world where speed and shape vary infinitely (astronomer as he was).

In what may be seen as extended interpretations of Galileo on mathematics, he argued that mathematics is the means of simplifying the complexity that is nature, of abstracting what is physically undefinable. Without mathematics how do we even attempt to paint a picture of the universe in 'words' (the words of mathematics were all geometrical then), the prose of the natural languages being too wordy for the purpose. There is only one way of decoding nature's existence, and there is zero margin for 'play of words'. Galileo's faith in mathematics was metaphysical. To him, we would be much worse off without the mathematical order of the universe; it is for humans to decipher that mathematics. In his words '*nature is inexorable and immutable, never violates the terms of the laws imposed upon her.*'

He believed that natural languages are best for scriptures because they allow for interpretations and contexts to be read by all. On the other hand, nature is the same for all, at all times, but it needs to be mediated to be 'read', which is done by mathematics. The conviction and gusto with which Galileo powered a mathematised science, a mystique developed around the idea of mathematics that transcended domains of knowledge and ignited a general academic rethink. In his book Turning Points in the History of Mathematics, Hardy Grant elaborates on the specifics – '*the clarity of ideas, the certainty of inference, characteristic of mathematical thinking became beacons … gave a model to those who would organise and expound their realms.*'

Four centuries later, nature, therefore humankind too, must be far more mathematisable.

**Research – Halo to hello**

Newer scientific discoveries are almost all mathematised, and descriptive scientific knowledge is rapidly shrinking in proportion. Prospective discoveries are anticipated to be more challenging for non-mathematizable hypotheses. Conventionally, this implies more accurate hypotheses and easier acquisition of all kinds of experimental setups, calling for a pat on the backs of a growing community of highly successful scientists, and engineers. However, on the ground, there is much more than this – data and mathematics are reading and generating certain patterns in the work of nature and indicating newer scientific laws and regimens.

Scientific research is on steroids due to (generative) mathematical modelling (and ever-multiplying computing power, by itself a mathematical marvel). And there are many faces to this research revolution – 'real world measurements negating the need for experiments and simulations', 'continuously learning, generative artificial eyes and minds replacing human observations', 'mathematical models that can secure patterns from real (Big) data (newer, older and highly diverse)', 'more organic collaboration of scientists and mathematical modelling', and 'degentrification of research, away from high costs, best minds, etc.'. This book is not the space to talk eloquently about how scientific research is mutating; several tomes around it are already on the shelves. Its relevance to the book is how research is getting intensely mathematised and degentrified in that process.

The recent achievements in complete genome sequencing of individuals exemplify how leading-edge research (and innovation) flows where mathematical models and computing powers are gorging on massive data. A new ultra-rapid genome sequencing solution by Stanford Medicine scientists and partners diagnosed rare genetic diseases well within ten hours – an unheard of diagnostic feat. Euan Ashley, Professor, placed it in perspective –'*this diagnosis (took place) in about the time it takes to round out a day at the office*'. It also reminds us that a few weeks for sequencing a person's genome is what most clinicians call 'rapid.'

Genome sequencing presents a person's complete DNA makeup, and it is a dream insight into all diseases rooted in our DNA. It means effective diagnosis, faster and more focused treatment, and much lower cost for a patient.

The mathematical genius in this solution is securing what is named a Long-read sequence; it reads the entire genome together. In contrast, the standard genome-sequencing techniques simplify the computing and mathematical threading by slicing the genome and then building it again after figuring out the particular DNA Base Pair details of the slices. Unsurprisingly, this whole-read method is about 12% higher than the average rate for diagnosing hard-to-pin diseases.

The other part of the solution was the fastest possible speed of data crunching. Genomic data had overwhelmed the facility's computational systems; in Euan's words, *'we had to completely rethink and revamp our data pipelines and storage systems (to efficiently crunch massive datasets).'*

Here is a hint on the size of the digital equivalent of a single person's genome data. Each of the 3.2 billion DNA base pairs in a human genome can be encoded by two bits—800 megabytes for the entire genome. But, for the sake of uncompromised accuracy, the sequencing is repeated, and the genome data for just one person can grow to be a few tens of gigabytes. Processing this much data implies more and more time. To overcome this problem, scientists have suggested that a person's data can be reduced to the differences with respect to a reference genome sequence, which is only a few megabytes.

Data, mathematics, and massive computing power are redefining the scientific research method that always started with stating testable hypotheses, experimentally testing the same (as a proxy for the real world), and succeeding or modifying the hypothesis and the experiment to ensure 100% confidence in experimental outcome's interpretations. Now, data replaces experiments, mathematics, the hypothesis, and specific mathematical formulations, the confidence required for universal acceptance. The catch is that data without a (mathematical) model is just useless. Of course, actual experiments may be used to prove it beyond all doubt, just as it happened to Einstein's gravity predictions.

To cite another example of how data is being used to create drugs that pose prohibitive ethical, cost, time, knowledge, and technology barriers, such as finding alternate drugs for multidrug-resistant pathogens or 'smart' pathogens that host drug resistance outside their cellular DNA (the usual target of drug researchers). In fact, many times, and increasingly so, we need comprehensively new chemicals (molecules) for drugs that 'surprise attack' the targeted pathogens; physically creating and testing new drugs of this kind will be beyond real possibilities.

AI is emerging as the panacea for drug development. It is being used to train neural networks to test structurally new molecules for their interaction with the targeted bacteria or virus without physically creating those molecules and the experimental setup. AI platforms can test (screen) over a billion molecules for their impact on a given pathogen, as opposed to around a tenth of that volume by traditional physical setups. Briefly, AI is a product of 'data and mathematical models'.

Something fundamentally more is happening due to AI – data is now enough to generate and test hypotheses. This is the 'Big Data' revolution. Fed with massive longitudinal and latitudinal data, 'AI software' read their patterns, if any, and articulate them in ways it is trained for. In essence, it is seeing the 'whole' data at a glance, repetitively, from multiple designs and perspectives. We do not have to look for models to start with (to be true, no hypotheses) or possible correlations to be expected in the real world represented by the data. We essentially require the best possible computing power and basic statistical tools to start with to see patterns and hypotheses those patterns may represent in the context of the data's real correspondence.

Chris Anderson, the editor-in-chief of WIRED magazine 2008, published a visionary article, *'The end of theory: the data deluge makes the scientific method obsolete.'* He predicted that in the era of petabyte data (1 petabyte is 106 gigabytes) and supercomputing, experimental pieces of evidence to accept or reject hypotheses would be unnecessary and wasteful. He had emphasized that at the petabyte scale, information transcends simple three- and four-dimensional taxonomy and order, evolving into dimensionally agnostic statistics.

In the times ahead, the traditional, hypothesis-centred scientific method of discovery would be one way to study our world. He declared that forgetting taxonomy, ontology, and psychology is essential because the crucial aspect is not why people do what they do but the fact that they do it. Tracking and measuring their actions with unprecedented fidelity is the key. The 'track and measure' he referred to is the Big Data – the real-life, extensive, actual record of the apparent behaviour of the target object(s) or situation(s).

Thus, first, we should let mathematics loose on the data and then look at making sense of the mathematical peek into the data to develop valuable insights, theories, and discoveries. Being able to parse through the massive amount of information in its entirety multiple times is an entirely new way to look at the world. He emphasised that in the era of petabytes, the numbers speak for themselves. The Petabyte Age is different because more is different, and AI represents a fundamentally new way of discovering science. Just data can throw up a probability distribution of the relevance of the most plausible theories explaining over served patterns. Prior training in an AI platform may be optional. Even simulations are desirable but not required, as AI is developing to dig deeper and identify patterns, holes, dimensions, and anomalies that humans cannot get to; there is now a 'third way' of research beyond experiments and simulations. However, collating, combing, and concluding Big Data is not just about mathematics and brute number crunching power. Eric Schmidt, the former CEO of Google, once made an insightful remark regarding the use of AI in science. He mentioned that by appropriately managing things with sound regulations and providing adequate support for innovative AI applications, we can expedite the process of scientific breakthroughs that would otherwise take decades to achieve.

The harshest of the evolving truth for the scientific community – lots and lots of data (infinite actually) can recreate all the science knowledge to date, hypothetically, at the least. Nature's ways never change; science is 'nature as read and explained'. Yes, the data capture of infinite nature is the unsurmountable gap.

Obviously, a better quantitative sense of the infinite nature will settle the idea of (one) Big Data that is powering the new hope and promise; there are infinite 'Big Data'. The upcoming Square Kilometer Array (SKA) Telescope, an intergovernmental project located in Australia and South Africa, is to be a source of data collection to keep peeking into it to know better some of science's most complex and humankind's oldest secrets, including potential existence of intelligent life elsewhere in the universe. Peter Quinn, the Executive Director of the International Centre for Radio Astronomy Research, provides an analogy to illustrate the vastness of the Big Data from the SKA: this telescope will produce an amount of data in a day equivalent to the entire planet's data output in a year.

Mention must also be made of the significant discoveries that Big Data is to make – hypotheses are much closer to real to start with, the extensiveness and quality of hypothesis testing is a few orders of magnitude more, and the testing itself is near fail-safe. New theories leave little out, and incrementalism in research and discovery may well be over.

Is there a spanner in all this? Mathematics! For all of us, and the leading scientists and mathematicians. For the lesser mortals like us, the mathematical and computational tools used to analyse Big Data are mostly opaque in their assumptions/limitations, outcomes, and the inherent limitations of mathematical formulations. Not difficult to imagine is also the real chances of the ill-use of such data-mined research.

All this brings us to the most enabling of all transformations in research; all of us can use Big Data analysis to read sense in the already projected patterns and seek further analysis for patterns that one can imagine to make sense. The power of diversity, untrained expectations and goals, lived and unique experiences, unfettered wants and needs, as well as the imagined world we wish to dwell in are all sources of 'good and grand' research in the times ahead. This also includes the liberation of scientists from pre-research grants and permissions. There are what Eric Schmidt calls 'self-driving labs' – AI-powered virtual research labs that people can hire. Last but not least, it portends the rise of AI as research-sovereign, AI analytics suggesting and accepting research agendas for the future.

Research is following the master – as the theoretical sciences expand, research mathematises.

**Innovation – The new socio-economic marathon**

What research is to science, innovation is to technology, and there is a data-led radical shift in the process, value-addition, and mathematisation of innovation. There is little to elaborate beyond the discussion on (Big) data-powered research, except that intensely (Big) data-driven organisations will likely be a few times bigger in customer service and profitable operations.

We shall now explore some well-known examples of innovations on the back of Big Data. Well-known examples of innovations on the back of Big Data.

Google is the world's most 'data-denominated' business, and the largest user of Big Data technologies. It showcases all that is possible with digital, mathematised, Big Data innovations – high-performance organisation design and culture, rapid product redefinition, evolving revenue models, quality post-sales support, and a formidable product pipeline. Yet, all these do not imply that Google gets it all right and thriving. There is much more to valuable innovations – correctly reading the several 'Big Data', faithfully translating the same into products/services, positioning it right, and keeping up with continuous innovation to beat the competition.

Google's pioneering work in advertisement-backed business model is nothing more than applied mathematics. It bets on generating large and varied data on its user's behaviour, continuously developing and tweaking analytical models to mine the user data, optimising product features to service a large population at manageable costs. To best sustain this organisational DNA, it also became 'employee-centred' in several path-breaking ways (all of that is part of history now).

Google's Universal Translator, duly unveiled in 2023, is definitely a technology marvel; it can seamlessly live translate video content across 300 languages with a lip sync of the live video for each translated language outcome. This is real due to Google's translation algorithms (codified mathematical models, such as statistical analysis) and Big Data on specific language usages. In 2023, Google has 17 years of longitudinal data on actual, nuanced translations of sentences by native speakers. The criticality of Big Data is evident in how the Translator already works much better for 'high resource language' such as German, for there is more than enough volume of written work in the language to train the AI.

Pertinently, the unsatisfactory translation performance in low-resource languages (used strictly in a limited sense of low availability of literary and communicative reference text data) must improve with better algorithms. In the final analysis, Big Data has the upper hand in the quality and expanse of AI applications. Peter Norvig, a distinguished researcher in human-centred AI, may have the last word on the place of algorithms (software mathematical models), *'essentially all models are wrong, but some are useful.'* And in case this sounds unreal, Chris Anderson's words clarify the importance of Norvig's approach. Why not explore the possibility of having computers rapidly learn models from data instead of humans painstakingly deriving models through extensive contemplation? In the end, all knowledge should be thoroughly logicalisable and mathematically modellable.

However, mathematical modelling of making sense of the Big Data embedded in algorithms does help compensate for the lack of data, and this ability is also increasing with the better fit of existing and newer mathematical models.

In Google Translate, algorithms are improved through 'synthetic parallel data', deploying lessons from same-family translations as well as multi-lingual translation, better identification of 'noise' or missing genres in texts used for training AI, better combination of different dialects of languages, and more. The awesome aspect of the Translate algorithms is that it ‘thinks language’, it is very highly trainable – given comparable Big Data, it would translate Hindi to Farsi as confidently as it would translate Arabic to English.

Amazon, Netflix, Uber, Coca-Cola, McDonald’s, Zomato, Starbucks, and MasterCard are just a sample of companies treading the same path as Google.

However, it is the Jack Ma founded Ant Group in China that mirrors Google’s data-stamped existence; he created a bank without any capital investment as a bank. It turned out so massively successful that it was listed as the most valuable debut on any stock exchange; it was too good to be true and eventually scuttled by the government. Its online bank, MYbank, brought credit to small and micro businesses, traditionally out of favour with banks. To be true, it is just the most successful example of the ‘fintech’. There is a global revolution in financial inclusivity, enabled by intelligent and automated systems that permeate the entire business ecosystem for collecting Big Data – real-time, authenticated, and transactional data from financial actions, social media posts and interactions, customer and product profiles of businesses, etc. Big Data affords a holistic approach to lending personalised economic and financial solutions.

In China, small business borrowers would apply for loans through their smartphones, just a few clicks at that, and receive cash almost instantly if approved (a large majority of the applicants). No meeting bankers, volumes of financial records, or referees. It took no more than three minutes. And yet, the default rate was under 1%, and the cost of loan processing was just half USD. The online application and risk-management system collated and processed over 2,500 dimensions of information on borrowers in those three minutes; yes, such intensive and intrusive ‘legal/formal’ access to private details of a private business is worthy of attention.

China's social credit system is a source of 'credible' information on the bankability of loan seekers, one of the reasons why creditworthiness could be swiftly decoded. Social credit is a kind of national 'trustworthiness' rating system for individuals and corporations. Initiated in 2014, it envisaged a six-year plan to build a system to reward actions that build trust in society and penalise the opposite. It is also uniquely insightful because social credit is built upon two kinds of information – traditional financial creditworthiness and 'social creditworthiness' (this is based on information from a larger swathe of everyday living.) In an article for Mint, Jun Lou of Bloomberg reported a case involving 'social credit data'. The article highlighted the potential difficulty a small-business owner might face in securing a loan due to a drop in their social credit score, stemming from something as trivial as failing to return a borrowed umbrella. Digitally collecting such information in a public database is just another instance of what makes Big data so big in impact.

The bottom line – healthy profits for MYbank. And higher top line and bottom line for the borrowing small businesses; millions of entrepreneurial dreams turning real. A true positive sum game. Innovation is the strategy now! Innovation is not just an ideal goal. Innovation is the business, not the products or services; the latter two are just targets of innovation.

Big Data is re-architecting innovation in a few crucial ways by bringing in a world of unstructured and legacy data (on paper, images), 360° data, contextual auto-recommendations, almost eliminating the cost of prototyping, co-creating with users, etc. It is mass-scaling innovation, reducing collective waste, and fuelling wealth democratisation.

Big Data is now an organisational soft infrastructure, like human resource qualities, culture, practices, value systems, and ethical codes. It is critical for all kinds of organisations, not just businesses and governments. It consists of all recorded and recordable actions of all kinds of ‘persons’ – individuals, families, communities, businesses, associations, government institutions, and now machines (mobiles, computers, IOT, autonomous objects, robots, bots). It also must include new knowledge created by Big Data itself!

**Everyone is a (net) producer – An existential imperative**

Research and innovation serve as the cornerstone for a better tomorrow. The de-gentrification of research and the ‘business of innovation’ rely on all of us to play our part in creating a truly sensible and sustainable future together. The more individuals involved in the realms of research and innovation, the better for everyone. Fourth Industrial Revolution (4IR) technologies have not only united the world into a virtual village but have also democratised access to the best socio-economic soft-infrastructure, including education, health, and ‘ease of business’. Humanity has never before extended such a welcoming hand to all. This marks the most notable difference between 3IR and 4IR – the top-down versus bottom-up nature of the two.

Most notably, 4IR ‘softens and virtualises’ the value creation and production fabric. Economic resources and opportunities seamlessly expand, capitalizing on and cultivating individual productive instincts and cultural propensities. Strikingly, the DNA of 4IR is as biological as it can be – it is a double-helix economic miracle that combines two distinct models: the proto-industrial system (involving large-scale production but without the 'factory model,' powered by independent 'Own Account Enterprises’) and the intelligent-industrial system (a post-factory model with globally networked 'Own Account Enterprises)."

Undoubtedly, the unfolding of the Fourth Industrial Revolution (4IR) may still be far off, but it awaits the effective reinvigoration of 'Own Account Enterprises'—where every adult serves as a producer, contributing economic value. No one is left behind and designated solely as a consumer; rather, every adult becomes an economic sovereign. Ensuring economic surplus for every adult is the wellspring of dignified living, for even the poorest of poor assumes multiple social roles—spouse, parent, child, sibling, neighbor, and citizen—all social constructs requiring some level of financial means to fulfill these social identities.

Indeed, guaranteeing human dignity stands as the most revolutionary promise within the realm of possibilities offered by the 4IR. The vision of a world with 8 billion dignified humans is beyond our current imagination. It encompasses the broad socio-economic integration of all humanity, bridging divides between rich and poor, rural and urban, men and women, developed and developing nations, and other economic faults. To the extent that the 4IR represents a highly quantifiable transformation, achieving dignity, for one and all, can be mathematically structured.

4IR technologies possess the potential to address the needs of the ever-growing global population of 8 billion. They can transform this burgeoning populace into a virtuous cycle of socio-economic advancement for all, wherein every individual's life contributes to a better collective future. We are witnessing the most fertile substrate for ensuring dignity for all.

The concept of dignity is inherent, yet its universal recognition has been a relatively recent, notably enshrined in the 1948 United Nations Universal Declaration of Human Rights. However, in current times, the concept of human dignity is under severe scrutiny, being challenged and violated. Reports of abuse, violence, discrimination, humanitarian crises, and authoritarianism are common across nations. Nevertheless, human dignity encompasses far more than the absence of these adverse conditions. We are gradually understanding how deeply it is rooted in the economic empowerment of individuals. We must rectify this situation or risk living in the indignity of a subhuman existence. Worse, our acceptance of indignity for the majority might become an irreversible and unredeemable condition.

It is important to remind ourselves that governments, unfortunately the only functioning human collective, cannot solely sustain the dignity of the middle class, let alone provide adequate support for those living in poverty and deprivation. Developed nations already struggle with unbridgeable fiscal deficits, causing their per capita social investments to dwindle. Many developing nations have already reached fiscal collapse. It is noteworthy that, as we write this, the Australian state of Victoria, recognized for its social liberalism, has announced the withdrawal of its bid to host the 2026 Commonwealth Games, citing insufficient financial resources".

Humanity must conspire to leverage technology as never before. Let us collectively hope for the emergence of Human Businesses – next-generation enterprises dedicated exclusively and profitably to serving the common market of 8 billion people (similar to the European Union common market). These businesses would focus solely on producing products equally valuable to all 8 billion individuals, eliminating the existence of 'bottom-of-the-pyramid' products. Considering this prospect, among the first such products could be a same quality mathematics education, ensuring uniform high-quality K-12 mathematical outcomes for every child and working-age adult.

Individual dignity cannot exist without economic dignity. Universal dignity requires mathematising humankind.

**A twist to the tale – We all use deduction, all the time**

Signs indicate that mathematical and logical thinking is inherent to humans as a priori knowledge – no specific learning or training is necessary to deploy it in everyday situations. Academic application, however, might require formal education. This innate ability naturally grows as we go about our daily routines. We are already somewhat mathematised, awaiting further honing and expansion through formalisation. The challenge lies in our dependence on the formal education system – curricula, textbooks, assessments, and teachers – which does not systematically introduce the thought process of deduction (and how it contrasts with induction, the scientific thought process).

At a rudimentary level, evidence suggests that we share a sense of quantity with certain animals. Most animals exhibit an understanding of their physical capabilities, for example, they displaying a sense of assessment of the length they can jump over with ease and do not attempt jumping over a wider drain. Similarly, research indicates that some animals, like crows, can differentiate among 1, 2, 3, and 4 quantities of something.

Deductive reasoning is an authentic, powerful mode of thinking about conditions and situations, consistently applied in our daily lives. For instance, based on the boss's past behavior of consistent lateness to meetings, I deduce that today's meeting would not be an exception. Hence, I might arrive late without consequence.

Observing that most questions in recent exams were from six out of ten chapters in the syllabus, I decide to focus solely on those chapters for my preparation.

Noticing a decline in orders for a particular product over four months, I conclude that the company needs to invest in new products. Recognizing data sciences as the fastest-growing career, I plan to transition to become a data scientist.

However, deductive reasoning is not always correct due to flawed premises and overgeneralisation. For example, Ill-informed premises can lead to incorrect deductions. For instance, assuming that all green-leaved plants need sunlight, and therefore a red-leaved plant does not, (and it could be kept wholly indoors). This overlooks the fact that need for sunlight is not solely due to colour of leaf, but due to the presence of the green coloured pigment chlorophyll in such leaves. The red leaves also have green pigment chlorophyll but that is masked by overwhelming presence of red pigments in those leaves.

Misguided assumptions also affect decisions. For example, we are familiar with the fact that objects in an open space gradually cool down to reach the ambient temperature due to the dissipation of heat from the warmer objects. It is commonly believed that to maintain the warmth of a liquid for an extended period in a room, we need to heat it to its boiling point and then cover it. This is a typical practice to sustain warmth in an open setting. However, this approach is flawed. The speed at which heat is lost to the surroundings depends on the temperature difference between the liquid and the surrounding environment. The greater this difference, the faster the liquid cools to align with the ambient temperature. Moreover, the rate of cooling is significantly accelerated when this temperature gap is higher. It is essential to understand that all objects radiate heat in proportion to the fourth power of their temperature.

Overgeneralisation can oversimplify complex situations and can lead to inaccurate conclusions. Overgeneralised idea that rural folks are inherently 'simpler' and 'sorted', hence, Aman, a recent migrant to a city, must also possess the same attributes of being 'sorted'. Assuming that because ABC is deemed the cleanest and greenest city in the country according to a recent survey, my friends residing in a colony in this city must automatically be experiencing a high quality of life.

Deductive reasoning is akin to solving a puzzle. Like puzzles, it requires a trained and informed mindset to solve. The process involves collecting all information about how to solve  puzzles and applying it to the specific puzzle at hand. This similarity between deduction and puzzles makes deductive reasoning a personal skill. Notably, detective work heavily relies on deductive reasoning.

Generally, deduction involves recognizing and applying a set of broader truths, assumptions, or principles to specific situations in order to arrive at the most favorable decisions or actions guided by this comprehensive framework. The effectiveness and success rate of detectives heavily depend on the thoroughness and comprehensiveness with which they gather all types of information and evidence, without prejudice to the perceived value of the information. Subsequently, they apply deductive reasoning to the facts and evidence to narrow down to the specifics of the case.

It is interesting that many of us exhibit a strong and predominantly accurate intuitive and commonsensical approach when responding to emergent situations. The utilization of a subconscious logical, deductive reasoning process is undeniable in such scenarios. This is why deductive reasoning is a form of thinking prowess that cannot be easily artificially created and routinised.

To better comprehend deduction, it is essential to juxtapose it with induction. In short, induction, often referred to as the scientific method, is the process by which research proceeds to discover new scientific knowledge. Hypotheses validated by adequately repeated 'specific experiments' are utilized as general principles (laws) of science. In a sense, induction follows a bottom-up approach, while deduction follows a top-down approach to accumulating knowledge.

However, it is through deductive reasoning, often termed the 'mathematical route,' that many scientific mysteries are uncovered. For instance, the expansive quantum field theory, purportedly regarded as one of the most comprehensive physical theories of all time, might await the development of the precise mathematics needed to unlock its secrets. Robbert Dijkgraaf, a mathematical physicist and the minister of education, culture, and science of the Netherlands (appointed in 2022), strongly advocates for the omnipotence of mathematics in understanding nature. He asserts that the workings of the universe follow an ordered and uniform mathematical structure, taking a bolder stance by suggesting that a proper mathematical comprehension of quantum field theory could potentially provide solutions to numerous unresolved physics problems.

In summary, mathematical reasoning—deduction—is a common practice among all of us. It is a matter of formally acknowledging, encouraging, and refining what's already a part of our lives. Humankind is already at a certain level of mathematisation, and mathematised mathematics autonomously raises that bar.

**Another twist to the tale – Mathematisation of 'social sciences'**

The conversation up to this point must not imply that the ongoing trend of increasing mathematization is confined solely to science and technology. Despite living in highly science and technology-driven times, our future is equally enriched by the mathematization of socio-cultural aspects of life and work. To this extent, STEM-focused research, innovation, and businesses position themselves as self-appointed guardians of humanity's well-being, yet many recognize it as a mere facade.

The increased mathematisation of social sciences research is not a new phenomenon. The utilization of mathematics by what we term 'social media' is widely known. It is notably sophisticated and continuously refined and restructured. For instance, platforms like Facebook employ Big Data and algorithms to dynamically tailor the display of pages and content to individual users. Each action of the user—clicks, likes, shares, friending, comments, and tags—contribute to and refine the Big Data related to the social behaviour of individuals, diverse communities, businesses (via their pages), and various other dimensions. These platforms utilize intricate, constantly evolving algorithmic tools, such as Affinity score, EdgeRank, Time decay, and Edge Weight.

Socio-cultural structures are incredibly diverse across societies, making it impossible to categorize and encompass any specific set of guiding principles for the mathematisation of social innovations. Comparisons can hardly be made between the pace and complexities of economic innovations. However, within the sphere of socio-cultural life, there exists a unique potential for a tectonic shift in the quality of our lives—mathematising humankind for true democracy.

Political philosophy, institutions, processes, and practices hold significant implications for societies. The mathematisation of governance institutions, predominantly in certain segments of the executive, often referred to as e-governance, was anticipated to drive us towards a more robust democracy. However, evidence from various parts of the world indicates a burgeoning executive that stifles the voices of citizens and opposition, conducts intrusive surveillance on numerous fronts, and fails to uphold airtight privacy provisions for citizens, community organizations, and businesses.

Mathematisation is sine qua non for true democracy, yet it would never be sufficient for a nation if e-governance were the singular focus of the mathematisation of the political society. The most groundbreaking consequence of mathematising humankind lies in the democratic revolution within each country. Furthermore, mathematisation should guide us in establishing resilient, true democracies, a first-time opportunity for humanity. At the core of the unprecedented 'poly crisis' facing us lies the political and moral crisis—the failure of democracy.

Nurturing and sustaining true democracy involves numerous dimensions. Considerations include what constitutes true democracy (as opposed to our current state), why we might seek it (its potential advantages), and the feasibility of its realization in current times (whether such a political revolution is attainable). In this context, the mathematisation of democracy stands as our sole hope to position society and citizens at the helm and in control of governance institutions, policies, and laws.

Yes, e-democracy presents a distinct form of political organization within societies when compared to e-governance or e-governments. The latter, in its current form, restricts democracy by implementing unparalleled, comprehensive, and intrusive surveillance of citizens and societies in real-time. There exists an unhealthy and menacing imbalance in the information and information flow between citizens and government institutions and political parties. To establish the technological foundation of true democracy, the essential component is the mathematisation or logicalisation of core democratic processes. E-government will play a role, albeit in a manner that serves democratic objectives. *Mathematisation of socio-cultural aspects of life is set to deepen.*

**Linear Algebra – The easiest mathematics is firing Big Data**

Luck also favors mathematics, the most benign face of mathematics is occupying the centerstage of its applications. Linear algebra constitutes the fundamental framework in mathematical modeling for the development of AI applications. As its name implies, it operates within the realm of linear mathematics. Consequently, all equations studied in this field are linear, with variables used in their 'native forms'—specifically, to the power of 1. For instance, the equation

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$ represents a linear equation where $a_1$, $a_2$, ..., $a_n$ and b are constants, and $x_1$, $x_2$, ..., $x_n$ are variables raised to the power of 1.

It is algebraic, dealing with varying quantities and their interrelationships. For instance, the equation $5x + 2y = 7$ represents a linear connection between the variables x and y. It signifies that if x and y change in a specific manner, their values must adhere to this relationship to qualify as a solution to the equation. In this instance, the equation delineates a line in the x-y plane, with x and y as the two variables.

It involves finding what these quantities may amount to under given exact quantities of these variables; For example, consider the following system of linear equations:

$2x + 3y = 8$

$4x - y = 7$

We can apply linear algebra techniques, like elimination or substitution, to determine the values of x and y that meet these conditions. In this scenario, we can solve for y in terms of x using the second equation and subsequently substitute that expression into the first equation.

In fact, any situation involving more than one simple variable requires the use of linear algebra to mathematically articulate and utilize it as a generalized model. Proficiency in understanding and employing linear algebra tools forms the basis for effectively utilizing 'big data' to creatively tackle a wide array of scientific, technological, social, economic, and even political/governance objectives and challenges.

Linear algebra empowers us to envision, interact with, and manipulate n-dimensional scenarios—be it scientific, technological, social, and more—where each dimension represents different variables that collectively define these scenarios. This is an intriguing facet of mathematics in general, and specifically, of linear algebra. Most of us find it challenging to comprehend anything beyond 3-dimensional space or objects. However, some, primarily physicists, have approached visualizing a 4-dimensional combination of time and space. Yet, it's not far-fetched to imagine scenarios in n-space, depicted as ordered data involving a list of n variables.

In our pursuit to comprehensively understand and model increasing volumes of data, we frequently augment the number of variables during data collection. Greater intelligence necessitates the inclusion of more variables. Consequently, linear algebra is growing in potency and usefulness for constructing progressively intelligent devices and systems.

Linear algebra serves as a fundamental analytical tool for various systems experiencing an increase in embedded intelligence—such as in engineering (for example, analyzing the dynamics of flow in a network of pipes), economics (for example, understanding price, supply, and demand dynamics), science (for example, in weather forecasting), and consumer products (for example, in sound speakers).

Linear algebra allows us to easily comprehend and manipulate systems of equations featuring a large number of dimensions/variables, enabling us to solve them for practical and implementable solutions.

*Linear algebra and its applications*

Linear equations are notably effective in approximating real-world situations. One intriguing scenario involves quantities that require multiple dimensions or variables for complete definition or understanding. The simplest among these are quantities known as vectors—they possess one dimension as magnitude, akin to scalar quantities, while the other dimension denotes their direction of change. For instance, speed is a quantity defined by a single dimension known as magnitude. However, when considering speed along with another dimension—direction—it becomes velocity. Thus, velocity represents a vector quantity with two dimensions: magnitude and direction.

In linear algebra, vectors hold a fundamental position, playing a central role in various key concepts and techniques. Using vectors in linear algebra offers a significant advantage—they provide a robust and adaptable method to represent and manipulate complex quantities. Vectors can undergo addition, subtraction, scaling, and various transformations, which can be combined to create more sophisticated operations and structures. Consider an airplane landing, a scenario influenced by several vectors: wind, drag, wing and tail positions, along with Air Traffic Control (ATC) instructions guiding pilots on a specific heading (direction) for a set distance (magnitude).

A matrix, much like a vector, comprises a collection of numbers, while linear transformations encompass the set of all functions (functions that take vectors as inputs).

In linear algebra, matrices serve to represent linear transformations and are expressed through matrix multiplication. For instance, the rotation of a 2D image on a computer screen exemplifies a linear transformation, which can be represented by matrix multiplication.

Linear algebra holds significant connections to various areas of mathematics, notably including probability, calculus, and statistics, because it provides an efficient means to represent and manipulate data. Its role in statistics and probability theory is particularly crucial.

In statistics, data is frequently organized in matrices or vectors, where each row signifies an observation or data point, and each column denotes a variable or feature. Operations in linear algebra, such as matrix multiplication, are instrumental in conducting computations on these data structures.

In probability, regression analysis is a statistical technique used to model the relationships between variables. Linear regression assumes a linear relationship between the dependent variable and one or more independent variables. The coefficients in a linear regression model can be estimated using techniques such as ordinary least squares (OLS), which involves solving a system of linear equations—a core concept within linear algebra.

In calculus, linear algebra is used to study the functions of multiple variables and their derivatives; linear algebra facilitates the solution of linear systems of differential equations.

Linear algebra can also be used to study optimization problems, which involve finding the maximum or minimum value of a function subject to certain constraints.

It may be encouraging to realize that linear algebra is part of school-level mathematics. There is no reason for anyone to struggle with mastering linear algebra, except due to the quality of school-level mathematics education and a lack of rigour in the faith and belief of the educators in ensuring all students succeed.

In contrast, the application of mathematics in physics demonstrates a relatively stronger command over mathematical principles. While calculus computations might not be suitable for everyone, Linear Algebra, in comparison, is accessible to all—it's simpler. Even theoretical physicists tend to favor more familiar and 'simpler' mathematics. For instance, consider the Heisenberg uncertainty principle and the Schrödinger wave equation, both independent theories in atomic physics. They share a similarity in asserting that a more precise determination of an atomic particle's position would compromise the certainty of its momentum. Essentially, their mathematical formulations are alike. However, Schrödinger's equation gained more popularity as it relied on more familiar differential equations.

On the whole, the central position of simpler mathematics would spur the faster and wider mathematisation of humankind.

**Mathematising mathematics – The pearly gates of education**

This extensive topic is reserved for a later chapter in the book. The 'technology of education' stands as humanity's blind spot, revealing our struggle to comprehend the means to nurture an infant toward reaching even 'half of their human potential.' Unfortunately, it seems to reflect a race to the bottom, as exemplified when a U.S. president famously urged teachers to compete against Indian children in mathematics, despite the puzzling dilution observed in mathematics education in India.

We have misconceived education to the extent that Edtech is now hailed as 'Technology in education;' expanding technology's presence in a domain that is fundamentally social. Education relies on role model adults, peer interactions, conversations, observations, experiences, and the development of habits of both body and mind, such as reading and writing.

Furthermore, the intensifying institutionalisation of 'educating children' might be humanity's most significant misstep in the past 200 years. The advent of the Fourth Industrial Revolution (4IR) will inadvertently lead to the de-formalisation of education, returning it to the domain of parental guidance, family influence, and the broader societal community 'the village'.

The growing public apprehension towards AI, urging for regulation to 'combat it,' actually signifies our collective failure to grasp the core of education. It is high time we equip people to align with AI, prompting a redesign and revolution in education. To cut through the complex context, the educational revolution hinges on the lack of a just any one domain of knowledge, skill, value, or attitude that every school can effectively instill in all its children, without exception. Currently, we are attempting numerous initiatives, all falling significantly short of the mark!

There exists only one such domain – mathematics! Unfortunately, K-12 education shows the poorest possible record of achievement in mathematics. K-12 has yet to de-arithmetise mathematics, and view it as the language of the gods and the universe, the language intertwined with everyday life. The philosophical foundation supporting mathematics as a natural-like language dates back several centuries. David Sepkoski, from the University of Illinois at Urbana-Champaign, in his research on seventeenth-century mathematical philosophy, suggests that 'the epistemology of mathematisation is fundamentally linked to the epistemology of language.' Epistemology refers to the *'philosophical theory of human knowledge.'* For instance, the previously mentioned 'epistemology of language' could be interpreted as how we acquire and master a language.

Mathematising mathematics is non-abstracting mathematics. K-12 reassertion will start with mathematised mathematics.

Without further ado, let us just say that humankind is mathematising as K-12 remakes itself.

**Regulating AI – A lame debate without 'making men'**

Regulating the AI industry is an ongoing, contentious battle. Surprisingly, some industry leaders advocate for seeking regulation while simultaneously advancing their vision for AI platforms and products. It is premature to firmly adopt a position on regulation or delve deeper into its evaluation at this stage.

The crucial and fundamental issue concerning the progression or containment of AI is the current and future nature and level of organic intelligence. The interaction and relationship between humans and AI depend on the master's capability and how we strive to surpass and maintain superiority over AI—continuously growing to maintain our mastery. Human capabilities are boundless, and our only limits lie in how we educate ourselves, determining our individual and collective virtues and potential.

There is little debate when it comes to advancing the mathematisation of humankind, regardless of how or when we regulate the AI industry. Mathematisation also involves rejuvenating society by empowering its basic units—individuals and families. Society represents the unseen force and structure in our lives. It revolves around instilling the concept and guarantee of social welfare, allowing genuine democratic control over our shared destiny.

We must realise that debate on AI regulation lacks foundation, and is without considering how humanity will progress in the future. In fact, AI itself plays a crucial role in facilitating the ability of humankind to retain best control over AI.

Furthermore, without striving to ensure the mathematisation of the entire human race, the debate about regulation falls into the hands of a fraction of us who may have a better understanding of AI but cannot genuinely represent the best interests of all of us, or the potential collective advancements in the realm of AI.

It all comes down to 'what it means to be human', 'what defines our humanity', and how mathematics serves as the fundamental stepping stone to understanding 'what makes us human'. As a corollary, it raises questions about 'the essence of education', 'the connection between the education system and our humanity', and 'the role of mathematics in education'.

# Annexure

## Mathematisation Case Studies

*Trigonometry and Calculus*

Seeing is believing. Experiencing is truth. Let us experience 'mathematised Trigonometry and Calculus'.

Pertinently, among the more unique and important features of mathematics as a language and a domain of knowledge is that it has come to be so abstracted, regourised, and procedured for the sake of widest applicability that it is too expansive even for mathematicians. We, the authors, are not mathematicians, and despite that handicap we assert that of all kinds of researchers mathematicians follow the most specific interests (of course, that also implies that their contributions are most impactful for humanity.) Education of mathematics needs to be revolutionized.

The abstractness of trigonometry is widely acknowledged. John G Kemeny, a remarkable mathematician and computer scientist, questioned the relevance, stating that a considerable portion of his high school trigonometry course was dedicated to the solution of oblique triangles. However, he expressed that throughout his highly varied career, he never found an excuse to use these techniques and questioned the necessity for all high school students to devote several weeks to the subject.

On the other hand, calculus education misses out the beauty and the beast that it is. In the words of the mathematician Steven Strogatz, calculus insists on a world without accidents, where one thing leads logically to another. With the initial conditions and the law of motion, calculus allows us to predict the future or, better yet, reconstruct the past.

**Mathematisation of thinking**

Mathematisation of thinking is building natural-language-like competence in expressing real or imagined relationships of quantities. It would help to know that this mathematisation is best raised on high proficiency in the chosen language of academics. For, language mediates thinking, and a brain that is already accomplished in using abstract objects and constructs – words, syntax, grammar, semantics, morphology – is far better equipped to master another language. Mathematisation of thinking is to harness mathematics as a language to comprehensively and uniquely visualise and express situations involving quantities.

Interestingly, the benefits of mathematics as a language are well appreciated. But it quite ends there; it is not practised, not even among mathematicians. The reason for this divorced state of possibility and practice is very illuminating. Natural languages are so-called because learning them is simply by participation; just being all ears requires registering literal correspondence between the words and the objects/feelings they represent. The formal constructs of our first natural language – the mother tongue – are for literary writing capabilities, not for accomplished communicative writing or reading literature in that language.

However, a layer of formal learning is required for 'non-natural languages', or acquired languages, such as mathematics, art, music, dance, 'theatrics', games and sports. All these languages are somewhat innate, a kind of sense/knowledge, and thus, too personal, and need to be framed into a common framework for communication with others. For instance, music is so highly structured/framed that it is almost universal; good music is pure science (and mathematics). Music is all sounds that are pleasant to our brain; all else is noise; that is why music is quite universal.

Mathematics must also be structured to be effective as a means of communication. We already know how mathematics has the simplest demands and complexities as a language. There is little by way of convention in mathematics (for example, the way we write numbers draws Cartesian planes), and even the list of standard notations is not much. What is the twist in the story of mathematics that makes it a 100% precise language (every physical reality has only one mathematical expression) and 100% universal? The simple answer is what we call concepts and the rigid network and hierarchy of concepts. Learning and mastering these concepts needs formal education and their application in the routine.

Thus, the mathematisation of thinking boils down to intensively exploring the conceptual foundation of the various dimensions of mathematics. This implies a significantly toned-down role and place of 'rigorous, calibrated mathematics' in mathematics as a language. To be convincing, we have chosen two dimensions of mathematics – (secondary level) Trigonometry and (senior secondary level) Calculus – as case studies of conceptual exploration of mathematics.

We hope that a good read of the two case studies would see you falling in love with triangles and Calculus and imbue you with newfound lenses to critically and creatively quantify disparate everyday and professional contexts, thereby setting off a new relationship with mathematics and the world because the two are also the most challenging of mathematics in K-12. The contrast with the extreme abstractness of school mathematics should be apparent, and the place of 'process and proof-driven' scholastic mathematics may be respectfully questioned and revisited. We expect that the real nature of mathematics will be revealed and mathematics education will become the fountainhead of AI-age thinking humans.

# An introduction to 'Mathematised Trigonometry'

Trigonometry is placed on the cusp of secondary and higher secondary education and is literally the high point of secondary geometry, algebra, sets, and functions. It is also the best closure to the enigma that is triangles, in terms of the lion's share of geometry curricula up to secondary years. Yet, trigonometry education cannot be more recklessly designed and delivered.

Trigonometry is best introduced and internalised as a function, a special relationship between the angles and sides of triangles. Understanding trigonometry as a set of functions that dramatically simplifies visualisation and verbalisation of the three 'primary trigonometric functions' – sin(e), cos(ine), and tan(gent) – and their multiplicative inverses – sec(ant), cosec(ant), and cot(angent).

Even better, senior secondary's nemesis – the inverse trigonometric function(s), especially when coupled with calculus – is a delight to be introduced as a function.

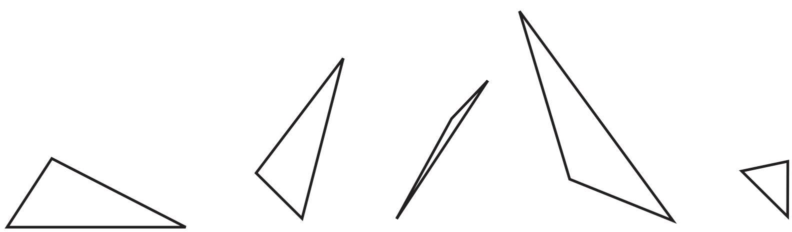
To be fair, relations and functions are often out of the secondary syllabi, and using these to explore trigonometry would not be possible without curricular reorganisation. But that reorganisation is anyway an imperative for another reason too – knowledge of sets is integral to counting, and quantification. Sets must be introduced in the pre-school years and better explored in the primary school years.

By the middle school years, interactions among sets could easily be studied at the basic level. Relations and functions are the operations through which sets interact. Relations and functions are the gateway to many mathematical foundations.

Besides, functions play a pivotal role in calculus. The latter would not be possible without the use of functions.

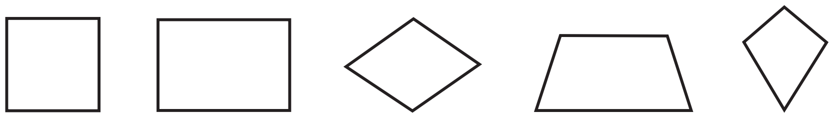
# Mathematised Trigonometry

Triangles, the simplest polygon we come across in everyday life and geometry have a wide variety of shapes and properties depending on the measure of the angles and the length of the sides. It would be hard to think of a 'standard' or more common kind of triangle, that is, a more common shape of triangles.

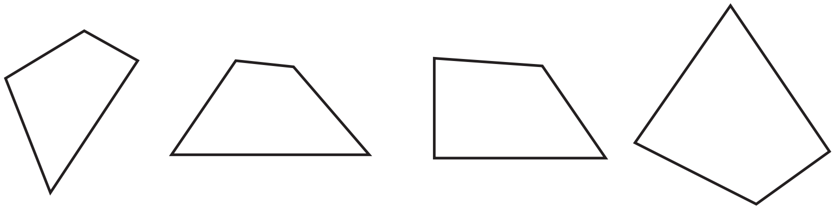


*Triangles of different shapes*

On the other hand, when it comes to the other polygons (with four or more sides), some shapes are so very common that they can be considered 'standard' or 'typical' polygon shapes. For example, parallelograms, squares, rectangles, rhombuses, and kites are the most visible four-sided polygons (quadrilaterals). Other kinds of quadrilaterals cannot be bracketed into having a standard or typical shape.

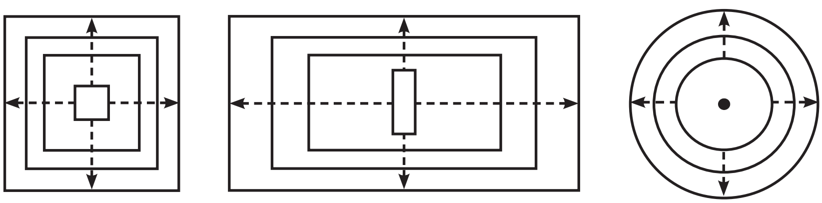



*Some standard quadrilaterals*



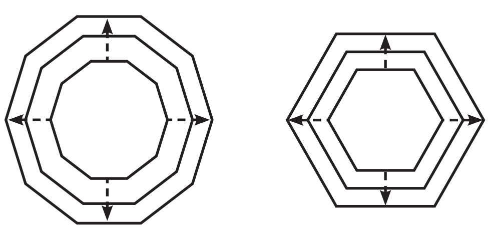
*Uncommon quadrilaterals*

The standard quadrilateral does not exhibit many variations, for example, squares and rectangles vary only with respect to the length of the sides. Thus, their shape changes to become bigger or smaller. Similarly, depending upon the length of the radius, a circle changes to become bigger or smaller only.



*Size of standard quadrilaterals and circles differ in length, not in shape*

The same is true for other kinds of polygons, such as pentagons, octagons, decagons, etc. In polygons, the standard form has sides with equal length. Such polygons are called regular polygons. There is no difference in the shape of the different versions of these standard (regular in geometry) polygons with the same number of sides. All pentagons, hexagons, and decagons, for example, are just bigger or smaller sizes of the same shape.



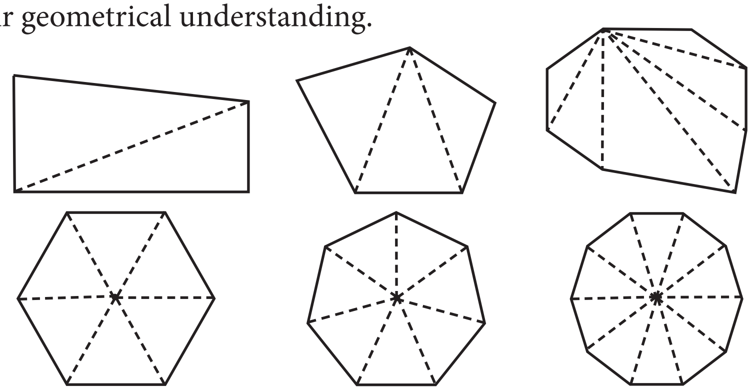
*Size of standard polygons differ in length, not in shape*

Triangles are unique polygons. They have innumerable variations.

**Welcome to the diversity and the beauty that is triangle**

The fact that triangles are polygons with the least number of sides and come in very different shapes is a boon in geometrical analysis. All kinds of polygons (four sides or more – whether regular or irregular) are geometrically studied by visualising and decomposing them into multiple interconnected triangles– their fundamental building blocks.

This also reinforces why we must study triangles in all their diversity, and they better not be reduced to any common forms for their geometrical understanding.



*Polygons are made up of multiple triangles*

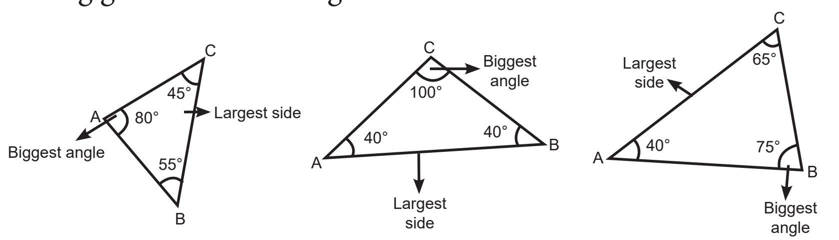
**The root of trigonometry**

Expectedly, there is a branch of mathematics that is dedicated to the study of triangles. It focuses on how the measure of angles and length of sides help compose infinitely unique triangles. To that end, it studies the relationships between the length of sides and the measure of angles of triangles.

**The saving grace**

Thankfully, despite the apparent diversity in the shapes of triangles, their properties reveal a remarkable simplicity when it comes to discovering patterns in the relationships between angle measures and side lengths.

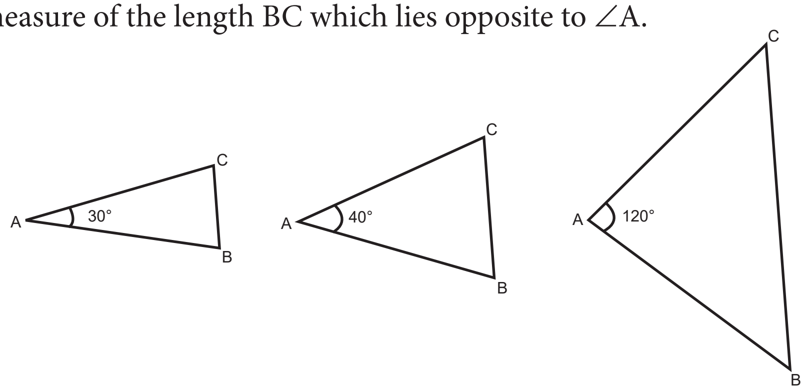
In triangles, a significant and indisputable relationship exists between the measures of angles and the lengths of the sides opposite to them. This fundamental connection acts as a saviour when tackling geometric challenges.




*Angle opposite to the largest side in a triangle is always biggest and vice-versa*

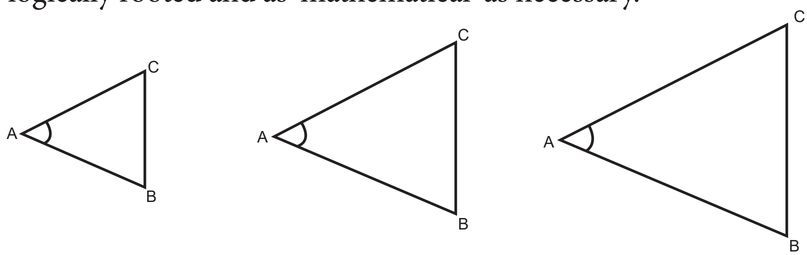
And even this relationship is specific – a given angle measure will not have a fixed length of side, it is limited to the fact that any increase in an angle measure will lead to an increase in the length of the opposite side (whatever it is).

As the measure of ∠A increases, so does the increase in the measure of the length BC which lies opposite to ∠A.




*With the increase in the measure of the angle, length of its opposite side increases*

There is another face of the angle and opposite side relationship – for the same angle, the opposite side could increase or decrease in length! Does it violate the nature of the aforementioned relationship? No. In fact, the finer, universal aspect of the angle and side relationship is in the form of angles and ratio of sides, rather than an angle and just a side. The other two sides of a triangle also change if the opposite side to an angle is changed. Thus, we would henceforth relate angles to sides. It must be emphasised that this is highly intuitive, practical, and visual correlation of angles and sides, it is not formal, but this is logically rooted and as ‘mathematical’ as necessary.




*Same measure angle with different lengths of the opposite side has different lengths of the other sides*

Indeed, in triangles the three sides and the three angles need to be studied extensively to understand its different facets.

Welcome to trigonometry, the domain of mathematics that helps us to measure all the six measurable dimensions of triangles – the three angles and the three sides.

**A note on learning about Trigonometry**

The documented roots of trigonometry can be traced back nearly 2500 years ago, and it likely has an even longer history that dates back further. Trigonometry emerged from astronomy in ancient civilizations as a practical tool for studying celestial objects.

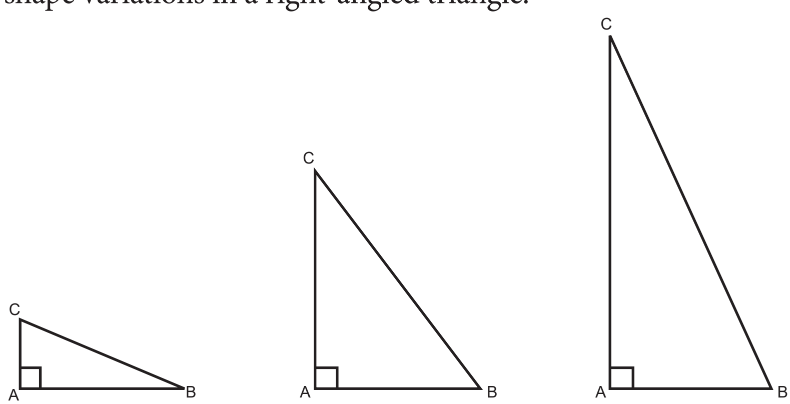
The geometry of celestial objects is 3-dimensional, not planar (2-dimensional, or what is called Euclidean geometry), and thus, the corresponding trigonometry is spherically oriented, not planar trigonometry.

At the best of K-12 geometry, spherical geometry (such as the shortest distance between two points on the surface of the earth such as the ‘straight flight path’ of migratory birds is not straight, it is an arc on the spherical surface of the earth.) is not part of curricula. And it need not be, planar geometry itself is a huge part of our lives, science and engineering too.

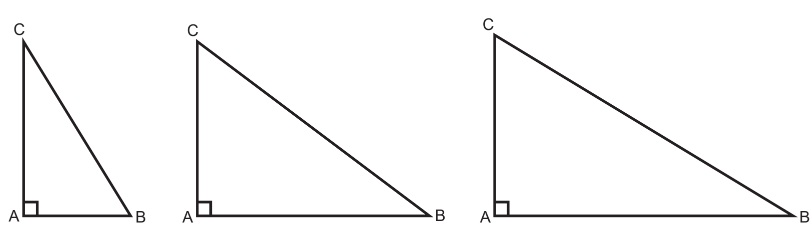
To the point, the foundational concepts in K-12 Trigonometry are made unduly complex by using spherical geometry for just studying planar trigonometry. For example, trigonometry was founded with ‘trigonometric functions’ in terms of arcs/chords of circles, but it does not mean K-12 education has to use the same foundations. As we will experience soon, functions (which are central to mathematics in all that is the real world) are a much better way to understand trigonometry.

Thus, we will explore the foundations of trigonometry from an easily visualisable and logically threaded narrative using functions.

The K-12 trigonometry also makes one more simplification – we study trigonometry for right-angled triangles only; this makes learning about sides and angles easier because the possible variations in the shape of triangles are dramatically simplified (still infinite in numbers). The following pictures show how there are only two kinds of shape variations in a right-angled triangle.



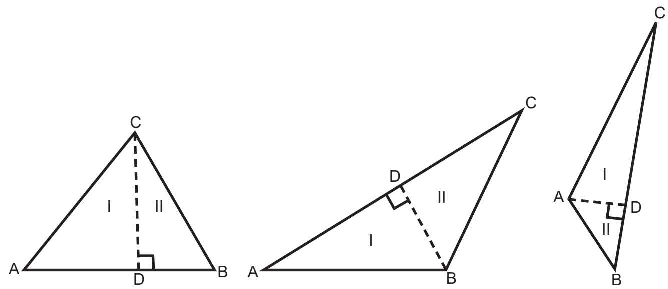

*Hypotenuse increases with the increase in height of a right triangle with same base length*




*Hypotenuse increases with the increase in base length of a right triangle with same height*

Remarkably, this simplification is similar to how regular polygons and circles are simpler shapes.

And the best news – there is no compromise in applying trigonometry as all triangles can be seen as composed of two right-angled triangles.

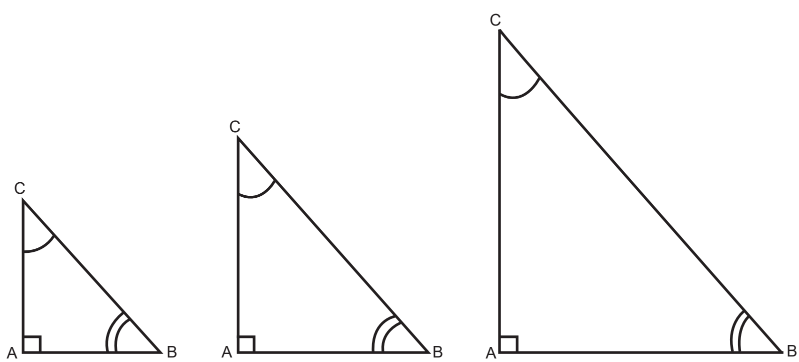



*Every triangle consists of two right-triangles*

**Trigonometry**

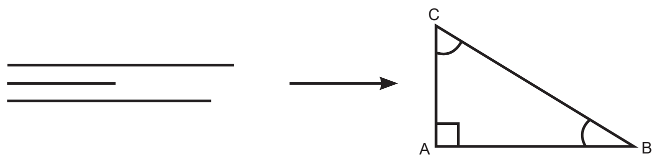
There is a very interesting fact about all the angles and all the sides of a triangle.

We cannot find the sides of the triangles even if we know all the angles – the same-angled triangles can have any-sized sides.




*Triangles with equal angles but varying sizes*

But the other side of the question is doable – we can find the angles of a triangle whose three sides are known. For example, if three lines of any length are given, then we can make a definite right-angled triangle with these three lines.



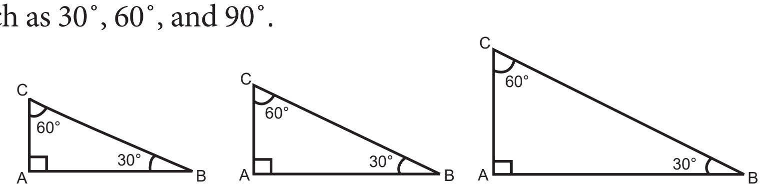

*Right-angled triangle can be created using three lines of any length*

Despite the fact mentioned above, the three possible scenarios with respect to triangles are - finding the sides when all the angles are known, finding the angles when all the sides are known, and when a mix of sides and angles are known and the unknown sides and angles are to be computed.

**First scenario**

We well know that when we know the measure of all three angles of a triangle we cannot make one specific triangle, there would be infinite combinations of valid three sides. However, the ratio of sides in such disparate triangles would be the same, because the angles in such triangles have the same measure (recall, angles and ratio of sides is the more universal relationship between angles and sides.) Thus, in triangles where we know the angles, we use the given angles to find the ratio of sides, the exact length of the sides could be any as long as the ratio of the sides is maintained for the given angles. When the three angles of a triangle are known, the closest we can come to knowing about sides is their ratio.

Let us closely observe the following triangles with known angles, such as 30˚, 60˚, and 90˚.




*Triangles of varying sizes with angles 30˚, 60˚, and 90˚*

Visibly, the one truth about the sides is that the ratio of any two sides of one triangle is equal to the ratio of the corresponding sides of other triangles. For example, $\frac{AC}{AB}$ in all three triangles would be similar.

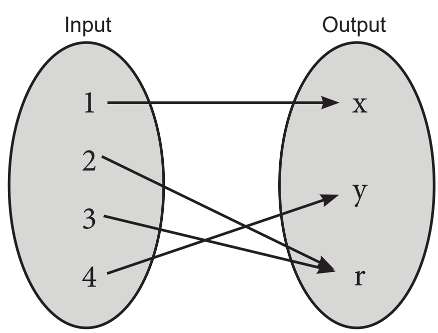
This relationship of a given angle and the ratio of sides in all triangles (made of the given set of angles) is formalised using the mathematical operation called function.

When the three angles of a triangle are known, the closest we can come to knowing about sides is their ratio.

Mathematics is a beautiful and powerful knowledge also because it invented functions. 'FUNCTIONS' take some input quantities and 'process' them to get an output quantity. They convert one kind of quantity into another. We would use functions which will convert angles into ratio of sides.

Mostly, functions are explicit and quantitative relationships between two or more quantities. A function defines how one quantity (the dependent variable) depends on one or more other quantities (the independent variables).

One of the distinctive features of a relationship that is a function is that for every set of input(s), there is only one and only one corresponding output quantity, and it will always be the same for those inputs. In other words, a function assigns a definite output value to each (set of) input value(s).




*A function is a relation that assigns to each input exactly one output*

The functions that relate the measures of triangles (which are only of two kinds – angles and sides) are called trigonometric functions; recall, trigonometry is the study of the relationships between the measure of angles and length of sides of triangles. Importantly, the trigonometric functions are applicable to all kinds of triangles, not just to the right triangles because all kinds of triangles bear a direct relationship between their angles and sides.

We are focusing only on the trigonometric functions applied to the angles of right triangles because it is simpler to study right triangles. Also, all non-right triangles could be seen and studied as two right-angled triangles. That's why, the school syllabus focuses on the trigonometry and trigonometric functions of right triangles.

Trigonometric functions take angles as inputs and produce the ratio of relevant sides as output. We have already explored how angles of triangles bear a direct relationship with the ratio of sides, rather than just the length of one side.

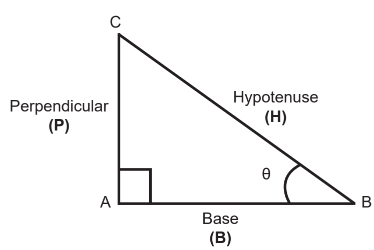
The term 'trigonometric functions' is indeed a more encompassing and intuitive name for these mathematical functions, especially when compared to referring to them as 'trigonometric ratios' (the more common name in school textbooks).

However, there is a natural query arises – what if the ratio of sides is known and the relevant angle needs to be determined?

We can use the inverse trigonometric functions to find the angle when we know the ratio of the sides.

Inverse trigonometric functions take the ratio of the length of sides of the triangle as their input and produce the relevant measure of the angle of the triangle.

Let us start with finding out how many ratios of sides exist in a right-angle triangle. Consider the following right triangle.




*A right-angled triangle ABC*

There are six possible ratios of sides in the triangle –

$$\frac{P}{H}, \frac{B}{H}, \frac{P}{B}, \frac{H}{P}, \frac{H}{B}, \frac{B}{P}$$

We need six different functions, which when applied to the angles of the triangle give us six ratios as their outcomes.

It is easy to appreciate that in all right triangles, one angle is always $90^\circ$, and any two right triangles are different only in terms of the other two angles ($\theta$, $\phi$ in the given figure). Thus, all the ratios of the length of sides of a triangle are linked to a distinct function with respect to an angle ($\theta$ or $\phi$).

Here is a table of six trigonometric functions of angle $\theta$.

| **Assumed Function Name** | **Ratio of sides** | **Trigonometric Function Name** | **Trigonometric Function (of angle $\theta$)** |
|---|---|---|---|
| First Function of angle $\theta$ | $\dfrac{\text{Side opposite to } \angle\theta}{\text{Hypotenuse of the triangle}}$ | **Sine** Function of angle $\theta$ | $\sin\theta = \dfrac{P}{H}$ |
| Second Function of angle $\theta$ | $\dfrac{\text{Side adjacent to } \angle\theta}{\text{Hypotenuse of the triangle}}$ | **Cosine** Function of angle $\theta$ | $\cos\theta = \dfrac{B}{H}$ |
| Third Function of angle $\theta$ | $\dfrac{\text{Side opposite to } \angle\theta}{\text{Side adjacent to } \angle\theta}$ | **Tangent** Function of angle $\theta$ | $\tan\theta = \dfrac{P}{B}$ |
| Fourth Function of angle $\theta$ | $\dfrac{\text{Hypotenuse of the triangle}}{\text{Side opposite to } \angle\theta}$ | **Cosecant** Function of angle $\theta$ | $\operatorname{cosec}\theta = \dfrac{H}{P}$ |
| Fifth Function of angle $\theta$ | $\dfrac{\text{Hypotenuse of the triangle}}{\text{Side adjacent to } \angle\theta}$ | **Secant** Function of angle $\theta$ | $\sec\theta = \dfrac{H}{B}$ |
| Sixth Function of angle $\theta$ | $\dfrac{\text{Side adjacent to } \angle\theta}{\text{Side opposite to } \angle\theta}$ | **Cotangent** Function of angle $\theta$ | $\cot\theta = \dfrac{B}{P}$ |

Similarly, we can link all the six ratios of length of the sides to the six trigonometric functions of angle $\phi$.

***The multiplicative inverse functions of sine, cosine and tangent***

The six trigonometric functions of angle ($\theta$ or $\phi$) have their inter-relationship in terms of the ratio of the sides.

The cosecant function of angle θ is the reciprocal of the sine function of angle θ. Similarly, the secant and cotangent functions of angle θ are reciprocal of the cosine and tangent functions of angle θ respectively.

Thus, the cosecant, secant, and cotangent are not the inverse functions of the sine, cosine, and tangent. They are the multiplicative inverse functions.

| Multiplicative Inverse Functions | In terms of Sides | Result |
|---|---|---|
| $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ | $\dfrac{H}{P} = \dfrac{1}{\frac{P}{H}}$ | $\operatorname{cosec}\theta \times \sin\theta = 1$ |
| $\sec\theta = \dfrac{1}{\cos\theta}$ | $\dfrac{H}{B} = \dfrac{1}{\frac{B}{H}}$ | $\sec\theta \times \cos\theta = 1$ |
| $\cot\theta = \dfrac{1}{\tan\theta}$ | $\dfrac{B}{P} = \dfrac{1}{\frac{P}{B}}$ | $\cot\theta \times \tan\theta = 1$ |

**Second scenario**

All angles can be found if the lengths of all sides are known. But how can we do it mathematically without actually measuring the angles? We can use the inverse trigonometric functions to find the measure of angles when we know the ratio of the length of sides.

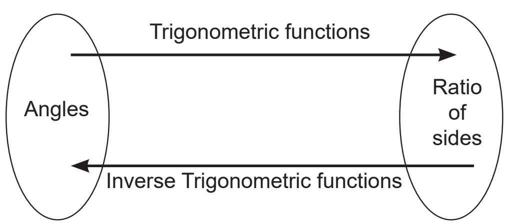
Inverse trigonometric functions take the ratio of the length of sides of the triangle as their input and produce the relevant measure of the angle of the triangle.

The inverse of the sine, cosine, and tangent functions is written as $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

| **Trigonometric Functions of an Angle is Ratio of Sides** | **Inverse Trigonometric Functions of Ratio of Sides is an Angle** |
|---|---|
| $\sin\theta = \frac{P}{H}$ | $\sin^{-1}\left(\frac{P}{H}\right) = \theta$ |
| $\cos\theta = \frac{B}{H}$ | $\cos^{-1}\left(\frac{B}{H}\right) = \theta$ |
| $\tan\theta = \frac{P}{B}$ | $\tan^{-1}\left(\frac{P}{B}\right) = \theta$ |
| $\text{cosec}\,\theta = \frac{H}{P}$ | $\text{cosec}^{-1}\left(\frac{H}{P}\right) = \theta$ |
| $\sec\theta = \frac{H}{B}$ | $\sec^{-1}\left(\frac{H}{B}\right) = \theta$ |
| $\boldsymbol{\cot\theta} = \frac{B}{P}$ | $\cot^{-1}\left(\frac{B}{P}\right) = \theta$ |

Trigonometry itself is quite a big deal, and inverse trigonometric functions simply scare us all, even in Grades XI-XII.

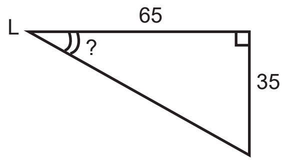
But, as we see above, inverse trigonometric functions are just another way of expressing the ratio of sides! Of course, the inverse also simply asserts that just like all functions have an inverse, trigonometric functions also have the inverse.




*Trigonometric functions – Angles gives ratio of sides;*
*Inverse trigonometric functions – Ratio of sides give angles*

***An illustration of trigonometric functions and inverse functions***

What is the measure of ∠L?




$$\text{Tan(L)} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{35}{65}$$

Strictly speaking, we have to find the measure of an angle given the ratio of the length of sides. The trigonometric functions – sin, cos, and tan – give us the ratio of the length of sides, given one of the acute angles of right triangles. But the need of the question is just the opposite – we have to find the angle given the ratio of the sides. This is a basic and classical case of using the inverse of a function – using a function to do the opposite of what it is made to do! Thus, if we use the inverse of the tan function, we will get to use it to find the angle for a given ratio of sides! We use an inverse trigonometric function here!

$$\angle L = \tan^{-1}\frac{35}{65}$$

**Third scenario**

In case the known is a mix of angles and sides, for example, two sides and one angle of a triangle are known, the computations do not change except for more arithmetical steps. The above two scenarios still hold true and adequate.

**Summing up**

The sine, cosine, and tangent (abbreviated as sin, cos, and tan) are three primary trigonometric functions, which relate the angle of a right-angled triangle to the ratios of two sides' length.

The sec, cosec and cot are the multiplicative inverse of the primary functions, respectively; what it implies is that sec, cosec, and cot are the arithmetical reciprocals of their respective primary function's ratios of sides. The inverse trigonometric functions do the exact opposite of the functions – they take the ratio of sides as input and give the measure of the corresponding angle.

# An introduction to 'Mathematised Calculus'

Calculus is real-world mathematics, far more than counting, numbers, arithmetic, (Euclidian) geometry, algebra, etc.; do not let your mind revolt against the statement, for example, when we count four apples, it does not mean the weight of the four apples are exactly alike (the weight of any one apple is approximate for the three). It is the most intuitive of all mathematical objects and concepts. No child can be struggling with visualising and verbalising calculus. Calculus education is all wrong; for the best of 'K-12 toppers', it starts and ends with limit and continuity.

A highly practical, real-world, and intuitive understanding of calculus is what we call 'mathematised calculus,' and that is what should be the content of calculus education in school years. Pertinently, it also best handholds us through the methods of calculus – mathematised calculus is the usual arithmetic, algebra, and trigonometry once past the foundational ideas and principles of calculus.

At its heart, calculus is about a world of derived quantities, the derivatives. There are many physical, real, critical entities, such as speed (and velocity), acceleration, electric current, power, electromagnetic force (comes alive due to magnetic flux), chemical product formation cycle, marginal cost, and utility, etc. that are not directly/physically measurable. For example, power is derived out of energy/work capacity and current is derived out of the amount of electric charge flow in a circuit in a given time; more importantly, both power and current are 'independently' meaningful and important quantities.

Expectedly, calculus is also about the opposite – the anti-derivative (integral), undoing a derivation process to get the quantity that was used to get the derivative. For example, the average velocity over a period is the anti-derivative of acceleration (itself a derivative of velocity) and volume is the anti-derivative of area. Similarly, derived quantities could be used to derive another quantity – the double derivative. For example, acceleration is the derivative of velocity, the latter is the derivative of distance travelled (over a period.)

*Derived quantities (derivatives) originate in change, detail the change.* All the aforementioned derived quantities, and all the others, have one thing in common – they are the rate of change of a 'changing quantity'! For a (continuously) changing object/situation, its rate of change is the real deal, the determinant of many things that matter about that change. For example, speed (at various instants of time) is the rate of change of distance traversed and determines the impact of accidents, the possibility of skidding at a sharp turn, etc.

Calculus is about measuring change; to be precise, measuring the change as it occurs – the change at different instants in relative terms (with respect to time or any variable quantity) to make more sense of the change. For example, knowing the amount of distance travelled is of little value until it is relatable to the time period of that travel.

*Anti-derivative describes the effect of change, not details of change.* It is the opposite – not the change at any instant, but the cumulative of the instant changes over a period of time (or any other variable quantity.) It is like a sum of the different 'instant, or infinitesimal' values.

*What is the nature of changing values?* Relevantly, for a changing quantity, an (infinite) series of the actual values of the changed quantities would need to be measured to understand it. But, a series of such numbers will be mathematically unwieldy and yet incomplete with respect to recording the changing quantities (we will soon exemplify this.) Changing quantities are precisely and comprehensively expressed using mathematical entities called functions, without explicitly listing every individual value. Briefly, functions are like input-output converters, quantifying a certain set of output for a set of input quantities.

Every 'uniquely varying quantity' is a 'unique function'. Every changing quantity is expressed as a function. Thus, we find derivative and anti-derivative of functions.

*Functions – The mathematical innovation to capture all instantaneous values*

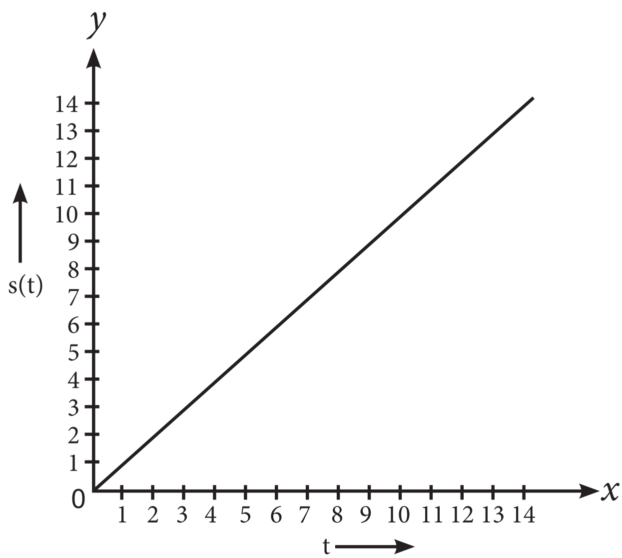
Imagine a bike whose speed at every one second interval is as under:

| **Time (sec)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Speed (m/sec)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

There are three obvious challenges with this instant values of motion:

- No pattern is directly visualisable (we need to graph it to really see the pattern in motion).
- There is no way to know the speed of the bike at any time other than given, for example, the speed of the bike at 3.5 seconds.
- To know more about the motion, tedious mathematical operations would be needed; for example, to know the nature of the acceleration of the bike, the acceleration values have to be computed for all 14 pairs of speed (start to the first second, first second to the second second, etc.).

However, we can overcome all the aforementioned challenges if we 'summarise' the speed and time relationship through a function. And that function, in this case, is 'speed = time', $s(t) = t$ (speed as a function of time is such that its magnitude is same as the magnitude of time itself, at all times between the start and the fourteenth second); it is more commonly written as $f(x) = x$.




*Speed-time graph*

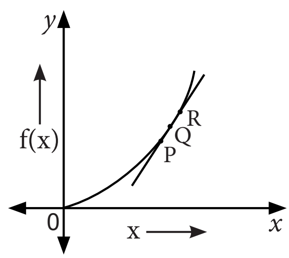
The entire world of mathematical measurement of change is founded on functions. The function gives the (infinite) series of instantaneous values.

When dealing with continuous quantities, functions are the primary mathematical tool for their representation because they can describe how these quantities change continuously. Continuous functions provide a powerful framework for modelling, analysing, and making predictions in various fields like engineering, science, business, etc.

Typically, situations quantifiable by counting represent discrete quantities, and those that need to be measured, or derived, and can take on an infinite number of values within a given range, are the continuous quantities. So, distance and time are continuous quantities whereas the number of students in a classroom is a discrete quantity.

However, functions can also be used for discrete quantification. Planning and controlling the efficiency of the production of limited-size batches of something is an example of creating functions for each discrete situation. For example, the number of batches produced, the number of items in each batch, the number of machines used, and the number of workers are typically whole numbers. Continuous functions need real numbers to be quantified.

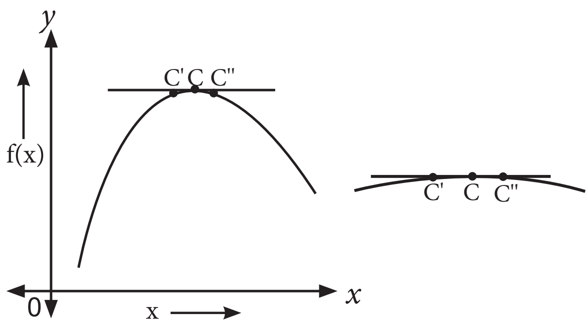
*How do we compute the rate of change to find derivatives?* The rate of change of a function at an instant, or condition, is the slope of its graph at that instant. Let us not forget that the rate of change varies in a changing quantity, thus, we think in terms of the rate at a point on the graph.




*The slope of the tangent PR at point Q is the rate of change of the curve/function at Q*

*Welcome the idea of limit!* Using the idea of limit, we make the slope of the tangent at a point becomes the best approximate value of the slope of the function at that particular point. Limit is the 'science' of infinitesimal quantities, a conceptual breakthrough in mathematics that laid the foundation of calculus – limit allows us to consider an infinitesimal part of the graph around a point that almost overlaps with a straight line tangent at that point. Recall, the slope of a tangent around a point is obtained by the simple rate formula.

Consider a horizontal tangent at point C and points C' and C'' close to C.




*The figure on the right is an 'infinitely enlarged' view of an infinitesimal part of the left curve*

As the points C' and C" get closer and closer to the point of interest C, the line becomes smaller and smaller while the slope of the line changes. When the points C', C, and C" are closest possible, the line becomes a tangent and the steepness (slope) of this line gives the best approximate value of the slope of the curve at the point of interest (point C). Another way to look at it is as follows.

Guaranteeing that limit does not go wrong! The concept of continuity complements the advantages of limit, by ensuring that the chosen infinitesimal part of the graph around the point to find the rate of change does actually represent the slope of the function at that point. Any sharp variation in the slope of the graph at that point is detected as a lack of continuity of the same slope at that point. In such situations, the limit of the function is said to not exist at that point, i.e., the slopes of the function just before and after the point are not the same.

*What is the greatest deal about calculus?* The derivative and anti-derivative of a function are the same for all the valid inputs for the function (the domain of the function). For example, the derivative of $f(x) = x^2$ is 2x, and it implies that the derivative at the point x = 2 is 4, and the derivative at the point x = 8 is 16.

There is more – solving calculus questions does boil down to knowing or computing the derivative or anti-derivative of the individual terms in the question and then following the usual simplification of the expression much like solving algebraic or trigonometric questions.

We are ready to consolidate this introduction of derivative, anti-derivative, limit and continuity as we read ahead the ‘mathematised calculus’ chapter.

**A note on calculus and 2023!**

The idea, joy, and applications get completely lost in calculus education due to its singular and rigorous rooting in the idea and computations, limit and continuity. 2023 is an -interesting milestone in calculus, and its education. In 1823, the French mathematician, Augustin-Louis Cauchy presented the text Résumé des leçons sur le calcul infinitesimal (‘Summary of Lectures on the Infinitesimal Calculus,’), his first book devoted to calculus, originally written to benefit his École Polytechnique students in Paris. The book is a remarkable work of conceptual vision and laid the foundations of the rigourised, formalised, particularised, and proceduralised foundations, concepts, and practice of using calculus.

The book had a sweeping effect on mathematics as a whole, and it massively guided and accelerated the development of ‘abstracted and methodised mathematics.’ However, it was meant to popularise and strengthen the correct applications of calculus among engineers and scientists. It was not meant to be used in introductory calculus education in schools, but that is exactly what happened, and an intuitive understanding of calculus was lost.

# Mathematised Calculus

**Change is the only constant**

Change is an inherent and unchanging reality of our world. Change refers to any alteration, modification, or transformation in the conditions of an object or situation (system). It can occur gradually and steadily over time without distinct breaks or interruptions or can be abrupt and sudden. Erosion and weathering of rocks, adaptation, and evolution of living things are changes that are very slow while volcanic eruption, earthquake, and landslide are examples of sudden changes. Continents, which seem fixed and immovable, are actually in continuous motion – a few centimetres per year. Middle school physics is built on the notion of constant acceleration (recall, F = m.a), but constant acceleration is a myth (even in deep space travel).

Let us not be deceived by many things around us that seem stable or unchanging, it is only a simplification of reality to make it easier to understand and compute at a preliminary, best approximate, and conceptually correct level. For instance, when we talk of averages, such as average speed over a 5-hour journey, we do not mean that the average speed was even momentarily an actual speed, it is just one good approximation of a range around which the speed lay in those 5 hours. The actual speed at different instants was not a constant that the average speed is computed to be.

The undeniable truth is that everything is in a continuous state of flux, change is the best hallmark of how our world is. To precisely and comprehensively understand our world, we need to explain and also measure the way change becomes evident in all things around us.

**To understand the world, we need to the understand the change**
Fortunately, our world can be visibly categorised into two broad kinds of objects –

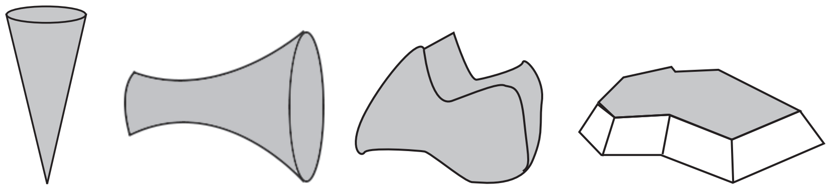
1. Objects that are stationary (buildings, trees, books, …)
2. Objects that move, or are in motion (motion may be the most ancient human fascination, starting with the motion of celestial objects)

Importantly, it is natural to think of change in stationary objects only through the lens of change in their position. But this would be missing the point about the change in stationary objects – change in such objects may also be in their weight, dimensional measures, the composition of matter in them, and others (to name just the quantitative descriptors).

Thus, the first characteristic of a change in an object is any kind of measurable difference in it between two instants of time, or any other conditions (such as in response to a change in pressure, temperature, etc.)

One of the special measurable differences of this kind is also the change in the dimensional measures – surface area, and volume (space occupied by a thing) of objects. For instance, imagine a rectangular packet of tea leaves tearing apart and a heap of the same tea leaves forming on the ground; the surface area of the heap and the packet would have changed (not the volume). And the usual geometric computations would not help in finding the surface area of the heap.


Similar computational challenges abound when attempting to find surface areas and the volume of 'curved objects', such as the following; (Euclidian) geometry does not work for curved surfaces and objects.

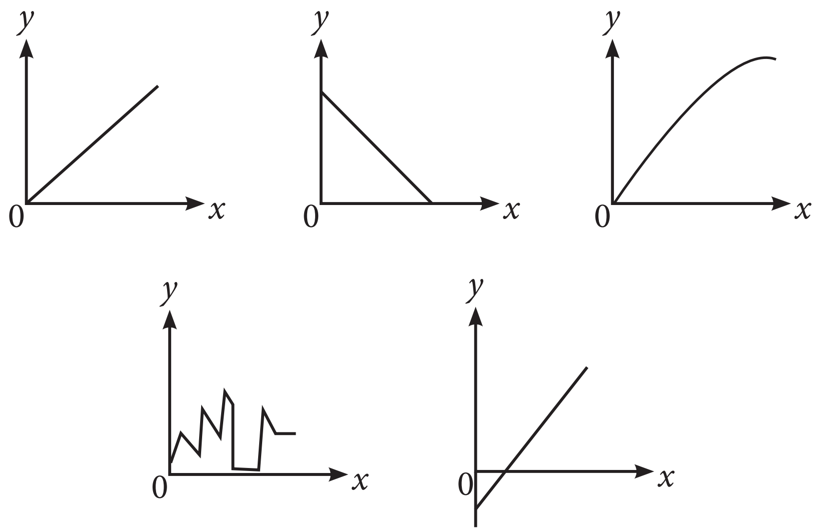


As to things that move, or are in motion, it is intuitive to think of the motion itself as representing change. Indeed, it is – change of position/place of the object in motion – but that is what motion is inherently about; there is no motion unless and until there is a change of position involved in it. However, a steady motion wherein the distance travelled, the time taken for that travel, and the direction are the same (if the direction is also relevant) over a time period, the motion would not be called to be changing. The motion would be said to be changing only when the direction and/or the distance travelled over the same time period changes.

Changing motion is literally the norm across the universe. All celestial bodies move along a curved path, may it be circular, elliptical or parabolic, hyperbolic, etc.; elliptical movements are the most common ones. This implies that the direction of motion of the celestial bodies constantly changes, and also implies that the distance traversed in fixed time intervals also changes constantly (in elliptical and parabolic motions).

Thus, change occurs IF a relationship between two, or more quantities is NOT steady; for example, if a body is moving in a way that the distance travelled by it over periods of time is different, then the motion is said to be changing.

Happily, graphs of the relationship between quantities are a very easy way to identify existence as well as the nature of change between the quantities. Here are a few examples of how graphs can show if change exists in a relationship, and how does the change look like:


y
x
0
y
x
0
y
x
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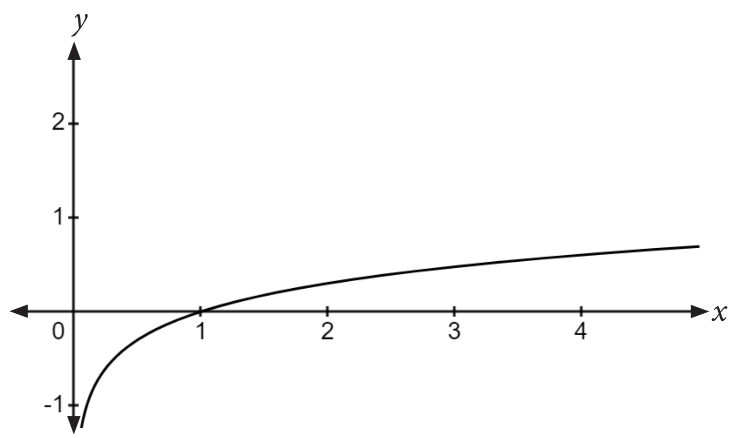
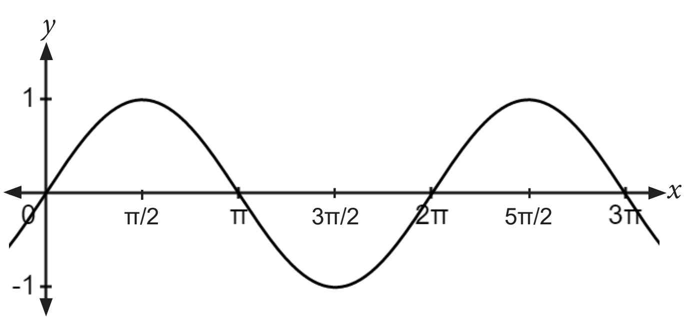
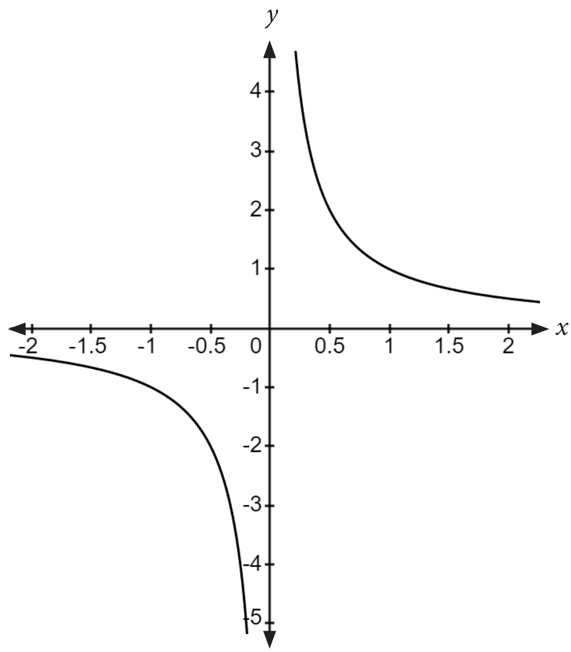
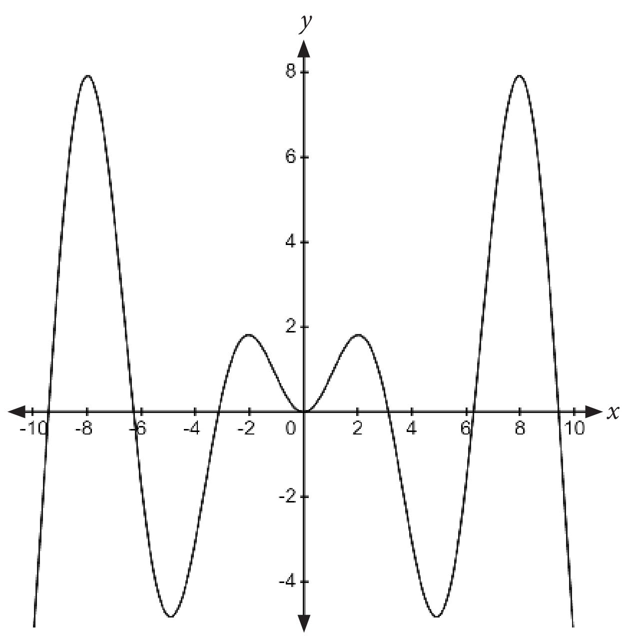
y
x
0
y
x
0


y
2
1
0
1
2
3
4
x
-1


y
1
0
π/2
π
3π/2
2π
5π/2
3π
x
-1


y
4
3
2
1
x
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
-1
-2
-3
-4
-5


y
8
6
4
2
x
-10
-8
-6
-4
-2
0
2
4
6
8
10
-2
-4

Interestingly, change can be registered only when it can be objectively measured quantitatively (for example, change in numerical values, 30°C to 45°C) or qualitatively (for example, change in colour or texture). However, change as a subject of mathematical interest implies changes occur when quantifiable relationships are not steady.

**Quantifying change – A series of instant values**

It is very interesting to realise that the biggest and quickest of changes are also in 'slow motion', steady, gradual, or what may be called 'a series of unique instantaneous values'. Time is amazingly divisible, and a change that appears in just one-tenth of a second is also slow and steady when looked at time frames that are one-hundredth of a second. The first one-hundredth second of the start of that change will register some kind of quantitative difference, the second one-hundredth second will bring in another quantitative difference (that will add to the quantitative difference of the first one-hundredth second), and so on.

The amount of change that is measured between the zeroth second to the one-hundredth second is the amount of change in the first one-hundredth second, and the amount of change between the end of the first one-hundredth second to the end of the second one-hundredth second is the amount of change in the second one-hundredth second.

At some slicing of time, all changes are just a series of many instantaneous values of change (all adding up together). Thus, to know a change, we need to study it as a series of instantaneous changes; the instants depend on the pace of the change, it could be the amount of change per minute, per second, per millisecond, etc.

However, the idea of instant has very interesting implications for actually measuring change. An instant means now and it is almost in some changes, the magnitudes between the beginning and end of observation could take an infinite number of possible values.

Such changes could be considered as an accumulation of an infinite number of infinitesimal or 'infinitely small' changes, occurring at each moment or instant. Such changes are so small that the change occurring between two precise 'moments' is nearly imperceptible. For example, the change in height of a child between his first and second birthday.

When we consider all the infinitesimal changes at all instants together, it creates the impression of an unbroken, continuous change in a system and gives a complete understanding of the behaviour of the change. When graphed, these changes are represented as a continuous line or curve.

To understand the events that are an accumulation of infinitesimal changes, and can change at any point and in any magnitude requires a language or framework that can effectively describe these dynamic and evolving systems. This mathematical language that represents continuous changes is a function.

**Function – Capturing realities in mathematical expressions**

Functions are mathematically expressed relationships of real-world situations, and they are such that for each change in any of the variables in the relationship, however small, a change is observed in some other variable of the relationship.

Continuous functions are a fundamental tool for understanding and making predictions about the behaviour of continuously changing systems in a wide range of fields. They enable accurate modelling, analysis, and optimization, making them essential for addressing complex and real-world problems.

To know more about continuous functions, refer to the Note 3 at the end of the chapter.

One of the real-world situations expressed as a function is the relationship between distance (D) and time (t) where both are variables and distance is dependent on time, i.e., D = f(t). Here, 'D is a function of variable t' means that there is a mathematical relationship that describes how D changes or depends on changes in t. So, the function 'f' takes the value of 't' as input and produces the corresponding value of 'D' as output. There is a unique value of D for every value of variable t.

Various mathematical functions define real-world situations, such as $f(x) = x^2$ is a quadratic function that represents a parabolic path, $f(x) = x^2 + 4$ is a quadratic function with a vertical shift such as energy levels or distances with a constant offset, $f(x) = \frac{1}{x}$ is a rational function that describes situations where one quantity is inversely proportional to another, $f(x) = \sin x$ is a trigonometric function representing a sine wave that models oscillatory behaviour, etc. It is written in a way that one quantity is seen as varying, or dependent on the way other quantity (independent) vary. For example, in the function, $y = \sin x$, where x is the independent quantity and as x varies the value of y (dependent quantity) varies. The notation commonly used to represent to describe functions is either y or f(x).

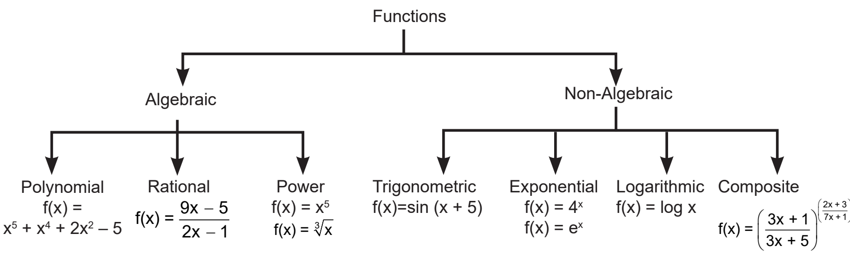
**Functions as input and output processors**

Input is a quantity that is 'entered' into a function. The quantity should be such that it is valid for the function, for example, the function $f(x) = \sqrt{x}$ is only valid for positive values of x. And, after processing, f(x) returns a value, the output. For example, if $x = 4$ then the output f(x) is $+2$ and $-2$.

The possible set of valid values of the 'input,' the independent variable, is called the domain of the function. The processed set of values of the output, the dependent variable, is called the range of the function.

**Finite set of functions**

Function $f(x) = x^2$ represents a parabolic function and function $g(x) = ax + b$ represents a straight-line, both being algebraic functions. However, $h(x) = \sin x$, expresses a sinusoidal wave and is a trigonometric non-algebraic function. We can categorise functions more elaborately as follows:




*Different categories of functions*

**Instantaneous value of a function**

Instantaneous value is essentially the value of the function at a specific point in time or space, taken at an infinitesimally small moment. For modelling and analysing various natural phenomena and real-world systems, we need to quantify the change in instantaneous values of the function.

**The challenge in computing change in instantaneous values**

The challenge in finding the change in instantaneous values or rate of something is the measurability of changes at that particular 'instant, or point/condition (for example, measuring distance travelled at an instant, i.e., measuring the distance covered for a duration that is nearly zero). The divisor in such computation of rate is nearly zero (we call these nearly zero quantities 'infinitesimal', which means infinitely small). Mathematically, such quantities/numbers can be visualised but any attempt to physically measure such changes is near impossible; imagine measuring the distance travelled by car in 0.001 seconds (0.001 seconds being the time assumed to represent 'an instant').

Thus, there are three challenges which we face while computing the change in instantaneous value or rate of something:

- The physical challenge to precisely measure a small quantity within a short time frame.
- The computational challenge in which the divisor is almost zero.
- The conceptual challenge that such small quantities do exist.

**Solving the physical challenge of instantaneous values**

Physically, it is impossible to correctly measure a quantity which is small in magnitude for a small measurement window. This physical challenge is resolved by using the idea of an indirect quantity, a derived quantity.

A derived quantity is a new, special quantity derived from another quantity (primary quantity). It is the quantification of some new aspect of a change in primary quantity. It is not a directly measurable quantity. It can only be computed using primary quantities.

Here are some examples of the derived quantities – power is the quantity derived from the primary quantity energy/work; force is derived from momentum; electric current is derived from electric charge; and electromagnetic force is derived from flux.

The derived quantity is called the derivative of the primary quantity out of which it is created.

**Derived quantities out of function**

Functions represent the world of relationships among quantities, and they are also the source of deriving new quantities or information from the primary quantities. This derived quantity or the new function obtained from the primary or original function is the derivative of that function.

The derivative of a function is the mathematical operation that works on a function to 'derive' indirect meaning(s) from the function. The primary meaning of a function lies in the relationships it represents between quantities; for instance, interest amount is the direct meaning derived out of the function that relates interest earned on a principal amount deposited in a bank for a time. From the derivative of this function, we can obtain indirect information, such as the instantaneous rate at which the interest is accumulating etc.

Derivative quantifies some new aspect of a changing quantity, for example, when we know the relationship between time and distance traversed by an object in motion, we can derive the speed and acceleration of the object in motion. The derivative represents the rate of change of distance (the 'something else') with respect to time. Evaluating the derivative at a specific instant gives us the instantaneous speed (the 'something') of the object at that moment. To be precise derivative gives a very specific new knowledge about something in change – the instantaneous value of 'something' with respect to the change in 'something else'.

The conceptualisation of the derivative of a function can be visualised as detailed hereunder:

- Something is continuously changing (slow, fast, regular or irregular …); for example, a car in motion continuously changes its position. It can change its position at a fast pace when on a highway or at a slow pace when in a traffic jam. Whatever the case is, it is continuously changing its position.
- The change in position is measured by a (physically measurable) quantity. Distance is that quantity which measures the change in position of any object in motion.
- The quantity that reflects the change in motion is distance. It indicates the change happening for any moving object.
- The rate of change of that quantity could also be changing. The change in distance in the various units of time could also be a variable, changing; for example, more distance is traversed in a certain period, as compared to an equal another period.
- The instantaneous value of the rate of change represents 'something else' which is another quantity related to the original quantity being measured. The rate of change of distance with respect to time represents speed, which may also change.
- The value of 'something else' is the value derived out of another quantity. Speed, which is the rate of change of distance with respect to time, is derived from distance.
- The derived quantity is called a derivative. Speed is the derivative of distance.
- The derived quantity is not a direct/primary measurable observation. Speed cannot be directly measured; in the way distance and time are measured.

- The derived quantity changes in tandem with the change in rate of change of the primary quantity. As the distance (which is the primary quantity here) varies over the different time periods of motion, the speed (which is the derived quantity out of distance) also varies.
- Theoretically, a derived quantity is also a function which can change with respect to any other quantity. It must be possible for the derived quantities to derive another quantity. The rate of change of velocity/speed is acceleration. Acceleration is thus derived out of velocity, which is in itself a derived quantity out of the changing distance.

**Mathematical expressions using derivatives (Differential equations)**
Recall that algebraic expressions are combinations of constants and variables that are put together using mathematical symbols, and algebraic equations are expressions that are set equal to zero. It is interesting to think that equations can also have expressions that incorporate changing conditions quantified through the rate of change. Such expressions are common, we mathematically express them every day and when used under scientific conditions, for now, they are called derivative equations or differential equations.

Wherever there are changing quantities in the 'equation' of a thing, the situation is mathematically expressed as differential equations. These equations can be used to configure everyday life to rocket science.
Refer to Note 4 to learn more about the differential equations.

**Anti-derivative of function**
As the name suggests, it is mathematically the opposite of the idea and the operation of the derivative. Let us construct the understanding of anti-derivative – one dimension at a time, out of the definition of derivative.

We know that a derivative is a function that gets created from another function. Thus, the anti-derivative must also be a function created by another function; (both 'input and output' of the derivative is a function, so the reverse view of derivative will also be 'input and output' as functions). For example, the function of speed gives the function of motion itself, the function of distance (motion is about a change of positions, distance), to be precise.

$$
\begin{tikzcd}
\boxed{\text{Function of Distance}} \arrow[r, shift left, "\text{Derivative}"] & \boxed{\text{Function of Speed}} \arrow[l, shift left, "\text{Anti-derivative}"]
\end{tikzcd}
$$

Next, we know that a derivative is a rate. The implication of being a rate is that it is a slice of the action, a 'part of a whole.' For example, the speed at an instant in a motion. What may be the anti (or opposite) of a 'part of a whole'? 'The whole' itself. For instance, what may be the anti-derivative of the (derivate) speed?

Let us take a second to scrutinise speed; it is the distance travelled in a unit of time (whatever be it), and it is a part of the (total) distance travelled. Indeed, a speed of 54 m/s implies that each slice of 1 second of motion is a distance of 54 m. The anti-derivative of speed is, in fact, distance, 'the whole'.

Of course, in the story of anti-derivative, 'the whole' needs to be identified because 'a whole' could also mean the universe. 'A whole' as anti-derivative needs to be specifically limited, and that is what is next explored.

We also know that a derivative is the value of something at an instant. What may be the opposite of an instant? A period of time, an interval of time. Indeed, anti-derivative qualifies (i.e., further explains) the quantity discussed previously (total distance travelled). The opposite of an instant is accumulated time or a period of time. And to the extent that anti-derivatives operate over a specified range of values of the changing quantity, the specification of that range is an input in finding anti-derivatives. Thus, the anti-derivative quantifies how much something has changed over a time period.

Anti-derivative is kind of a difference between two quantities – one at the start of the period or a state of things, and the other at the end of the period or state of things. Another way to see it is the summation of the changes or cumulative changes over a period or the range of change of another thing.

Common examples of the application of anti-derivatives to find out the amount of change in something (that is changing) are – from the anti-derivative of the function that represents the rate at which the tumour grows over time, the accumulated change in the size of the tumour inside the body of animals can be obtained. The anti-derivative of the function that represents the rate of population growth over time can be used to find the total population increase over that time interval. The impact of a head-on collision of two cars is a series of (very fast happening but) small changes in position after the collision. The anti-derivative of the function that represents the positions of the crashing parts of the cars can be used to provide the cumulative change or the displacement of the cars during the collision.

Here are a few more examples of the deployment of the idea and operation that is anti-derivative:

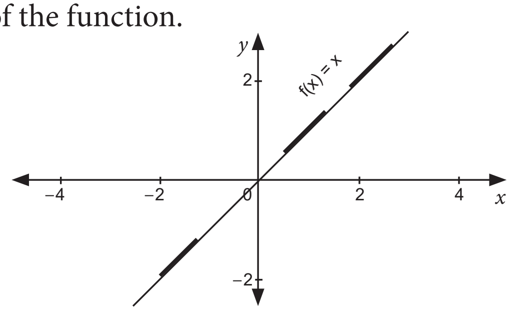
| Average/ Function | Anti-derivative | Explanation |
|---|---|---|
| Function | Average value of a function | Average value of a function over a range is the anti-derivative of the function. |
| Area | Volume | Volume is the anti-derivative of the area; over a dimension; similarly, area is the anti-derivative of one dimension. |
| Density function | Mass | Mass of an object is the anti-derivative of a density function (which is mass per unit volume) over a given volume. |

Let us find the derivatives and anti-derivatives of functions through their graphs. But before this, read about slopes from Note 1 and Note 2 given at the end of the chapter.

**Derivative of f(x) = x**

The derivative of a function at a point is the rate of change of the function at that point. Graphically, the rate of change of a function at a point is the slope of the curve at that point. A study of the slope of a curve indicates the derivatives of function at various points.

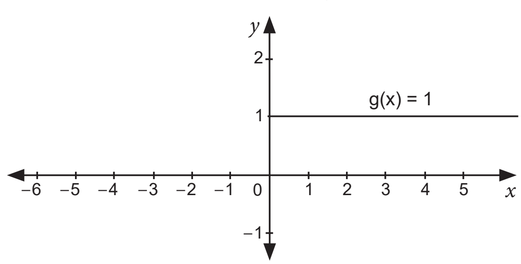
Consider f(x) = y = x which represents a linear function. The derivative/slope at each point within a domain of the function (the possible values of the input value x) having a linear graph is constant, that is, y changes at the same rate or constant value within the domain of the function.




*Various tangents on the linear graph of f(x) = x*

Since the slope is always a constant value for a linear function, the derivative of a linear function is a constant function and can be represented mathematically as g(x) = c, (c is any constant) and graphically as a horizontal line parallel to the x-axis.

For linear function f(x) = x, the slope is 1. Therefore, the derivative of f(x) is g(x) = 1, the slope of the given function f(x) = x, which is a constant function representing a horizontal line parallel to the x-axis. We will derive it mathematically later.




*Graphical representation of derivative g(x) of f(x) = x*

Since, the graphs of all linear functions are derived from the graph of y = x and possess the same properties, the slope/derivative of all linear functions is a constant.

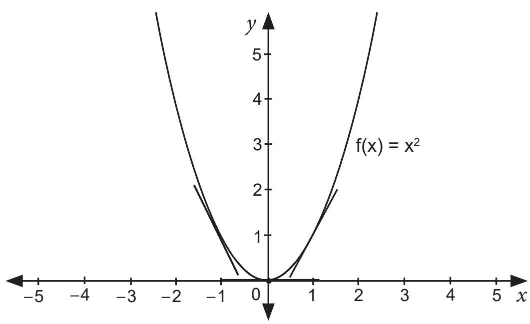
**Derivative of f(x) = $x^2$**

It is convenient to study the derivative of any function through its graph, and we will work with the graph of the given function f(x) = $x^2$ to find its derivative.

Recall, the derivative of a function is also a function and in its graph, the x-coordinate is the same as the function's x-coordinate and the y-coordinate is the value of the slope of the given function.

Thus, for the given function, we need to study the behaviour of the slope at different points of its graph to get the corresponding points to plot the graph of its derivative function.

Let us consider the graph of the given function f(x) = $x^2$, along with three tangents drawn at different points, to discuss the nature of the slope of the function.



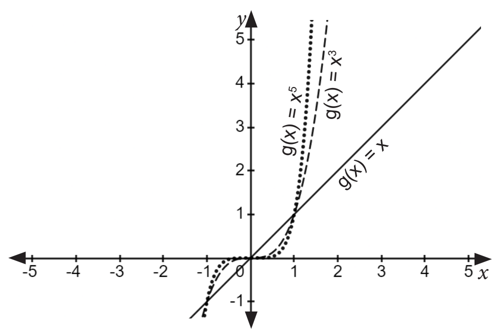

*Graphical representation of f(x) = $x^2$*

We can observe the following in the graph of f(x) = $x^2$:

- For negative values of x, the slope of the given function is negative and for positive values of x, it is positive.
- The slope is zero at x = 0, where the tangent is horizontal to the x-axis, in fact, it coincides with the x-axis.

- Based on these observations, two imperatives emerge for the derivative function of the given function, $f(x) = x^2$, where the derivative function's value is the slope of the given function (whose derivative is supposed to be explored).
- The derivative function passes through the origin, since at $x = 0$ the slope is 0.
- The values of slopes are increasing for $x > 0$ and decreasing for $x < 0$.

This suggests using the above arguments, the possible derivative function graphs could be any of the following $y = x$, $y = x^3$, and $y = x^5$.




*Graphical representation of possible derivatives g(x) of f(x) = $x^2$*

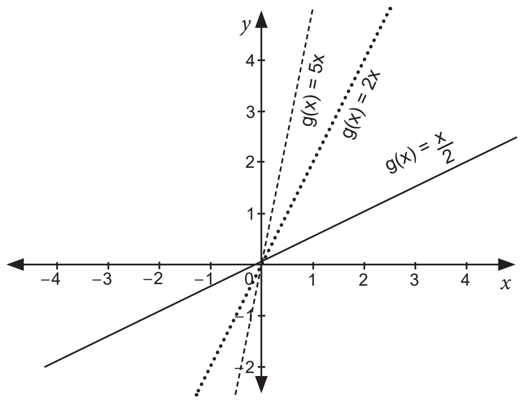
However, we also observe a gradual decrease/increase in the graph. This can be affirmed by the values which the function takes when the values of x are put in the function $f(x) = x^2$.

When $x = 0.5$, $f(x) = 0.25$
When $x = 1$, $f(x) = 1$
When $x = 1.1$, $f(x) = 1.21$ and so on.

Also, there are no sudden dips/rises in the values of the slope of the tangents considered in the graph of the function $f(x) = x^2$ because it is a continuous function. Recall that the steeper the tangent, the more is its value of slope. This suggests that the values of the derivative function would not be large for a small value of input. That is, the derivative function cannot be a curve such as $x^3$ or $x^5$, where for a small value of x we have a large value of the function.

This suggests a straight line as a derivative for $x^2$.

There can be many functions with this possibility that can be seen in the graph.

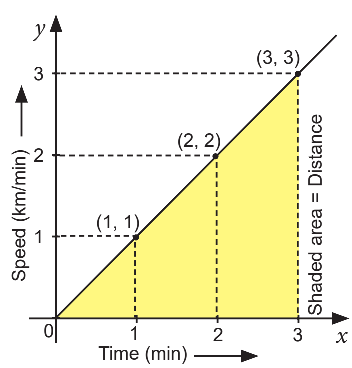



*Graphical representation of possible derivatives g(x) of f(x) = $x^2$*

The takeaway from this derivative graph is that the derivative of all quadratic functions is a linear function. We have logically derived the nature of the derivative function of a quadratic function. Later we will explore which of the above graphs is the actual derivative graph of the given function.

The derivative of $x^2$ is 2x. For now, let us generalise that the derivative of all quadratic functions is a linear function.

**Graphically exploring the anti-derivative of speed**




*Speed-time graph*

The graph represents a car in motion having a linear speed or we can say that the speed increases linearly with respect to time, i.e., at t = 1 min, it has a speed of 1 km/min; at t = 2 min, it has a speed of 2 km/min.

Since distance = speed × time, it can be best interpreted by the shaded region. The anti-derivative of speed is the area of the shaded region, which is the distance covered by the car in motion.

In fact, the anti-derivative of a function is a quantity that is the area under the curve of the function.

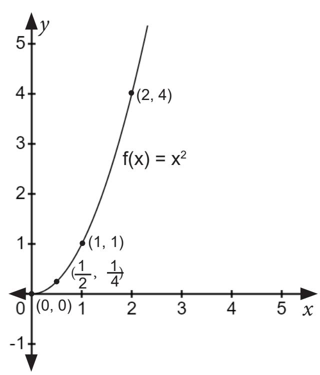
**Anti-derivative of f(x) = $x^2$**

We aim to find a function whose derivative is $x^2$. The anti-derivative function is one whose range values are the same as the various values of the area under the curve y = $x^2$ in some interval.

Let us assume the interval [0, 2]. The graph of $x^2$ in the interval [0, 2] can be obtained by making a table of the various points of x and the corresponding points of y.

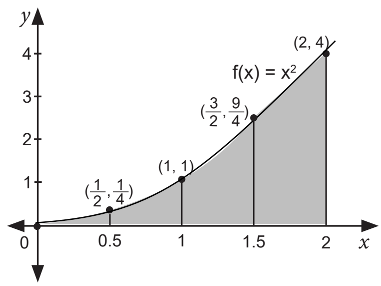
| x | 0 | $\frac{1}{2}$ | 1 | 2 |
|---|---|---|---|---|
| y | 0 | $\frac{1}{4}$ | 1 | 4 |

And, it can be shown as under.



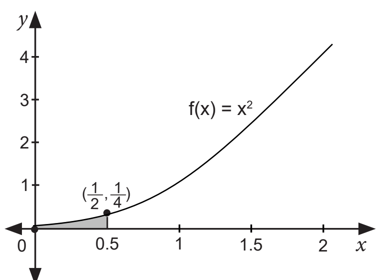

*Graphical representation of f(x) = $x^2$ in the interval [0, 2]*

Arithmetically, it is tough to find the exact area of the region that is curved. However, finding its approximate area is always possible and that basically serves our current purpose of broadly finding the nature of the function that is the anti-derivative of $x^2$. Logically, breaking the intervals [0, 2] into small sub-intervals would make better sense.

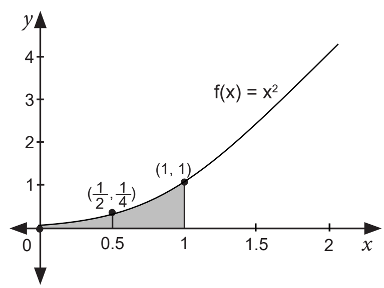



*Graphical representation of the area under the curve of $f(x) = x^2$ in the interval [0, 2]*

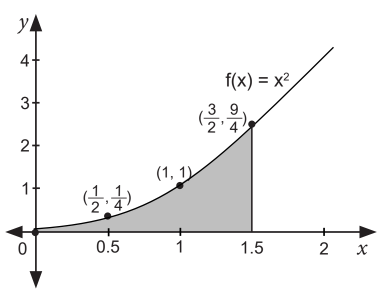
The area under the curve in the interval [0, x] can be approximated by a triangle with base x and height $x^2$.



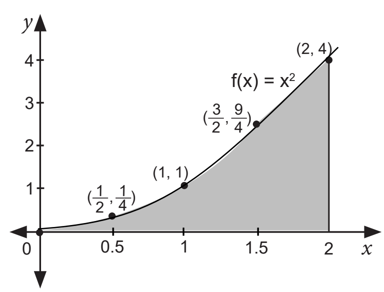

*(a) Area between interval* $\left[0, \frac{1}{2}\right]$




*(b) Area between interval [0, 1]*




*(c) Area between interval* $\left[0, \frac{3}{2}\right]$




*(d) Area between interval [0, 2]*

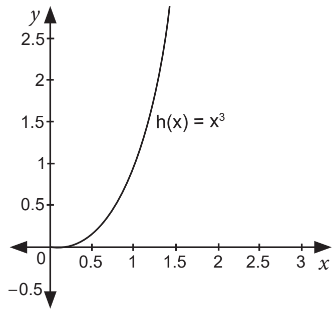
| Area in the interval [0, x] | Area of triangle $= \frac{1}{2} \times$ base $\times$ height | | | Coordinate for anti-derivative graph (x, area in the interval [0, x]) |
|---|---|---|---|---|
| $\left[0, \frac{1}{2}\right]$ | $\Delta 1 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4}$ | $= \frac{\left(\frac{1}{2}\right)^3}{2}$ | $= \frac{1}{16}$ | $\left(\frac{1}{2}, \frac{1}{16}\right)$ |
| [0, 1] | $\Delta 2 = \frac{1}{2} \times 1 \times 1$ | $= \frac{(1)^3}{2}$ | $= \frac{1}{2}$ | $\left(1, \frac{1}{2}\right)$ |
| $\left[0, \frac{3}{2}\right]$ | $\Delta 3 = \frac{1}{2} \times \frac{3}{2} \times \frac{9}{4}$ | $= \frac{\left(\frac{3}{2}\right)^3}{2}$ | $= \frac{27}{16}$ | $\left(\frac{3}{2}, \frac{27}{16}\right)$ |
| [0, 2] | $\Delta 4 = \frac{1}{2} \times 2 \times 4$ | $= \frac{(2)^3}{2}$ | $= 4$ | (2, 4) |

From the above calculations, we can deduce that the approximate area under the curve $f(x) = y = x^2$ corresponding to any interval $[0, x]$ is $\frac{x^3}{2}$.

On plotting the coordinates for the anti-derivative graph, we will get an approximate graph of the anti-derivative function of $x^2$, $h(x) = \frac{x^3}{2}$.

This is an approximate value, the real value will be less because the curve of the graph is concave. That is, instead of the anti-derivative being $\frac{x^3}{2}$, it can be $\frac{x^3}{c}$, where c is a real number.

For conceptual exploration, we can ignore 'c', and we can say that the anti-derivative function of $f(x) = x^2$ is $h(x) = x^3$. The graph of this is as under.



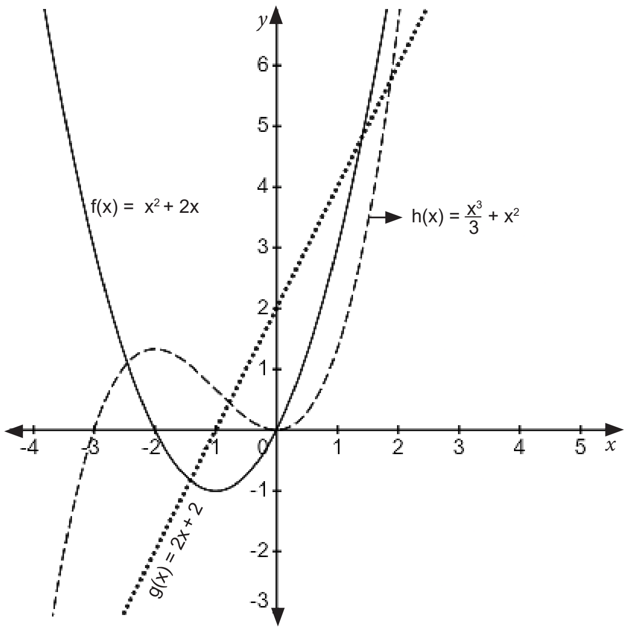

*Graphical representation of anti-derivative h(x) of f(x) = $x^2$ ignoring the constant c*

Thus, the anti-derivative of any quadratic function of the type $ax^2 + bx + c$ would be a three degree function. On actual mathematical computation, the anti-derivative of $x^2$ is $\frac{x^3}{3}$.

On combining the result for the linear and quadratic functions, the derivative of $y = x^2 + 2x$ is the sum of the derivatives of $x^2$ and 2x. The derivative of $x^2$ is 2x and that of 2x is 2. Thus, the derivative of $y = x^2 + 2x$ is $2x + 2$.

The anti-derivative of $y = x^2 + 2x$ is the sum of the anti-derivative of $x^2$ and 2x. The anti-derivative of $x^2$ is $\frac{x^3}{3}$ and that of 2x is $\frac{2x^2}{2} = x^2$.

Thus, the anti-derivative of $y = x^2 + 2x$ is $\frac{x^3}{3} + x^2$.



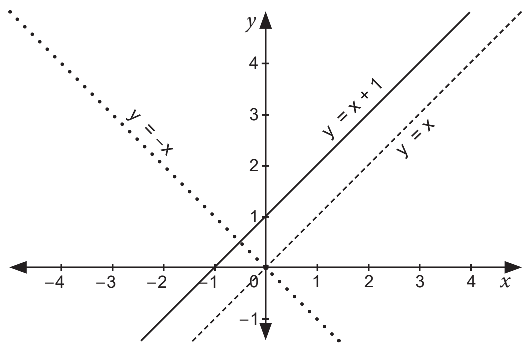

*Graphical representation of function $f(x) = x^2 + 2x$ and its derivative $g(x)$ and anti-derivative $h(x)$*

Now, we can find the derivative and anti-derivative of any function through their graphs. But there are infinite functions for infinite realities. Fortunately, math is the language to be used for expressing patterned conditions and relationships. And, inexplicably, it just so happens that just a few tens of patterns (i.e., well-defined, repetitive, 'universal' behaviour) lie at the core of the infinite realities. Expectedly a core set of functions – parent functions – do nearly express all kinds of situations.

**The parent functions**

When we graphically represent functions we can see that many functions' graphs look alike and follow similar patterns because these functions share the same parent functions. Parent functions are basic and the simplest form of functions. These functions serve as fundamental building blocks for constructing more complex functions. The complex functions from the same family of parent function can be easily recognised/graphed bearing the marked features of the parent function. Conversely, by taking the parent function's graph through various shifts, flips or stretches, all the functions within a family of functions can be derived.

There are infinite possibilities for creating a unique function from just one parent function. For example, y = x, y = –x, and y = x + 1 all represent a family of straight lines that can be seen in the given graph.

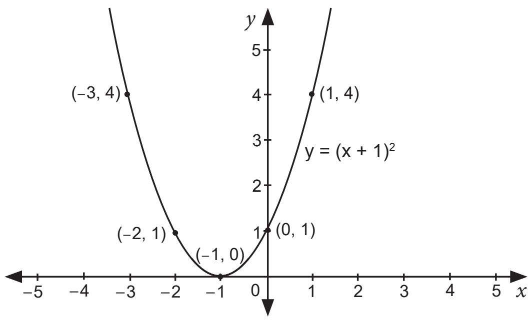



*Graphical representation of a family of straight lines*

If we observe the graph carefully, we will notice that the graph of y = x + 1 is shifted up by 1 unit from y = x and both have the same shape of the graph. Similarly, y = –x is a reflection of y = x about the y-axis. However, both the functions, y = x + 1 and y = –x look similar in a definite way. The transformations of the parent function in no way change the shape of the parent graph, and follow the basic characteristics of the family as defined by the parent function.

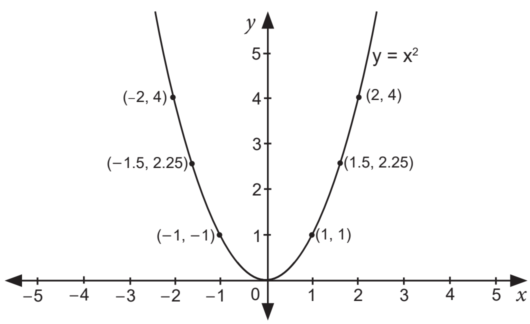
As another example, let us draw the graph of $y = (x + 1)^2$. Following the standard graphing technique, we create the following table of coordinates to plot the points on the graph.

| x | –3 | –2 | –1 | 0 | 1 |
|---|---|---|---|---|---|
| y | 4 | 1 | 0 | 1 | 4 |



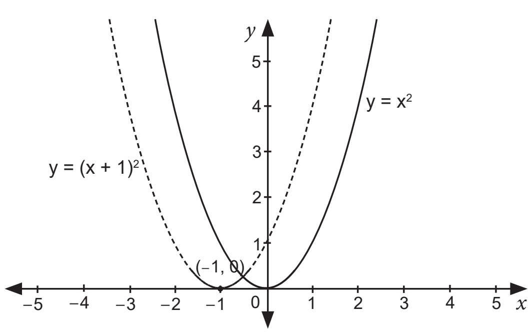

*Graphical representation of the function $y = (x + 1)^2$*

Let us also create the graph for $y = x^2$ using the same graphing technique.




*Graphical representation of the function $y = x^2$*

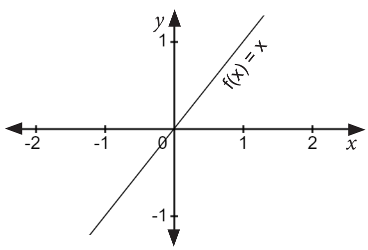
When the above two graphs are superimposed onto a single graph, they would look like:



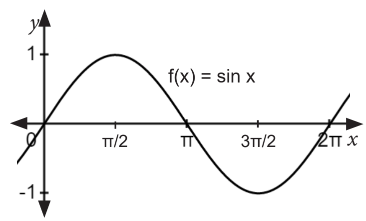

If we observe carefully, $y = (x + 1)^2$ is a parabola with its vertex at $(-1, 0)$. Its graph is similar to that of $y = x^2$, a parabola, with vertex at $(0, 0)$. Thus, knowing the graph and properties of $y = x^2$ can help us to know the graph and properties of the function y = $(x + 1)^2$ as well. Hence, y = $x^2$ is called the parent function of all other degree 2 polynomials, such as y = $(x – 1)^2$ and y = $(3x – 4)^2$. The graphs of both of these functions have the same shape, however, the vertex the case of y = $(x – 1)^2$ is (1, 0) and that for y = $(3x – 4)^2$ is $(\frac{4}{3}, 0)$.

We can combine the graphs of these parent functions to create a new combined function. Let us explain this with the help of a combined function f(x) = xsin x.

This function is a product of the linear function x and the sine function sin x. First, visualise the graphs of the individual functions.

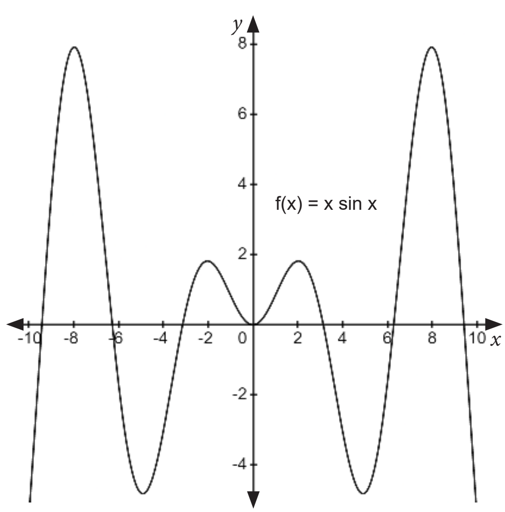



*Graphical representation of f(x) = x*




*Graphical representation of f(x) = sin x*

The sine curve oscillates between –1 and 1. However, its product with x will change the amplitude (the maximum height of a wave) of the combination function. This can be verified from the graphical representation of f(x) = x sin x.



**Limit solves the computational challenge of instantaneous values**

Let us get back to the challenge of finding the instantaneous value of a changing quantity and speed as an example of the same. We know 'the rate of change of distance', is speed, it offers the 'average' of the distance traversed over a period of time. The idea of 'average' is embedded in the idea of rate. The discussion eventually boils down to overcoming the limitation of the mathematical expression of average, which is the best mathematical means possible to find how much something changes due to the change in something else (such as time).

Recall, average speed is conceptually akin to a division expression (average is a kind of ratio that in its standard form is best read as division) and divisor of (almost) zero makes the quotient skewed towards being disproportionately bigger.

In the discussion on derivatives, we figured out that we can overcome this challenge by using the idea of an indirect quantity, a derived quantity, called the derivative; for example, the instantaneous value of speed (of something in motion), at a point, is derived from the way distance changes over time – the rate of change of distance travelled at that instant.

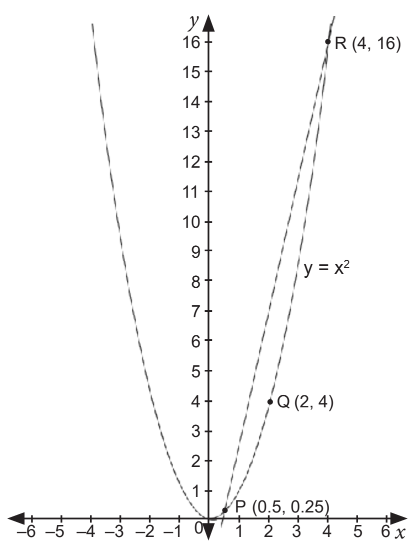
Conceptually this was a breakthrough, but computationally, finding the rate of change at an instant remained a challenge (when computing speed, instantaneous strictly means 'zero' time duration of the observation, and the distance traversed in that 'zero duration').

For finding instantaneous speed, we cannot have 'zero' time duration a divisor. The next best thing is to make the time duration so small that it is non-zero but tends to be as close to zero as possible. We need a non-zero divisor for the computation of instantaneous change to be possible, but to reflect instantaneous values, the nonzero divisor must be the smallest possible. When the independent quantity is non-zero, yet approaching zero, it is said that 'the limit of the quantity is zero'. This non-zero, but closest possible to zero approach, where the rate of change in the value of x is non-zero but near zero is called the limit of a function f(x).

Now let us study an example of a curve, parabola $y = x^2$, to see how we can get infinitesimal tangent on a point on the curve and use it to find the slope at that point on the curve.

We will find the approximate value of the slope on the point of interest Q (2, 4) by continuously reducing the distance between the two random points P and R (above and below point Q) on the curve to reach the closest to point Q to get the infinitesimal tangent.

The slope of this tangent would give the slope of the curve at point Q.



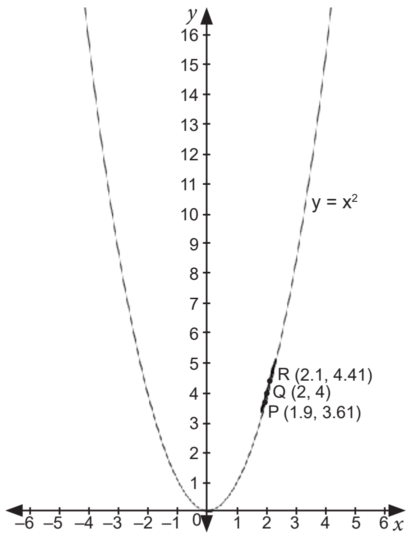

*Graph of $y = x^2$ with points P (0.5, 0.25) and R (4, 16)*

Draw a line through two random points P (0.5, 0.25) and R (4, 16) on the curve and calculate its slope.

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16 - 0.25}{4 - 0.5} = 4.5$$

This is a rough approximate slope of the infinitesimal tangent at point Q.

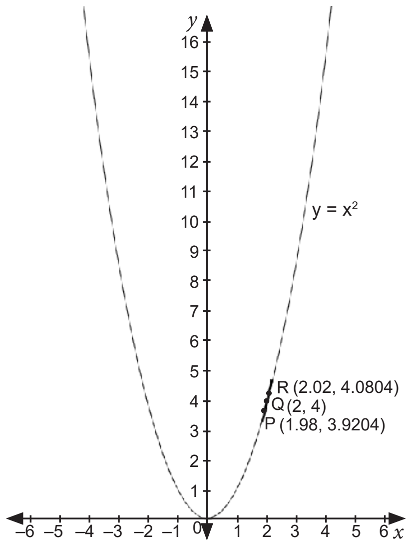
Similarly, take another set of points, say, P (1.9, 3.61) and R (2.1, 4.41), which are closer to the point Q (2, 4).




*Graph of $y = x^2$ with points P (1.9, 3.61) and R (2.1, 4.41)*

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.41 - 3.61}{2.1 - 1.9} = \frac{0.8}{0.2} = 4$$

Now as we move points P and R further close [i.e., P (1.98, 3.9204) and R (2.02, 4.0804)] to point Q on the curve to find a better approximate value of the slope of tangent on point Q.

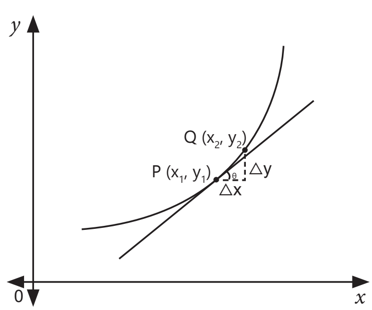



*Graph of $y = x^2$ with points P (1.98, 3.9204) and R (2.02, 4.0804)*

$$\text{Slope of PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.0804 - 3.9204}{2.02 - 1.98} = \frac{0.16}{0.04} = 4$$

As the points P and R get closer and closer to the point of interest Q, the line becomes smaller and smaller while the slope of the line changes. When the points P, Q, and R are closest possible, the line becomes a tangent and the slope of this line gives the best approximate value of the slope of the curve at the point of interest (point Q).

Another way to look at it is as follows.




*Graph of slope of a curvilinear function*

As seen, the slope of a line = $\dfrac{y_2 - y_1}{x_2 - x_1}$

For a curve, $x_1$ is taken very close to $x_2$ and $y_1$ is taken very close to $y_2$. Thus, $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ are very small.

Also, $\tan\theta = \dfrac{\Delta y}{\Delta x}$.

Thus, a slope of a line can be characterised using tan θ, which is the angle made by the tangent to the curve at the given point and the horizontal axis.

**Why is the limit so-called?**

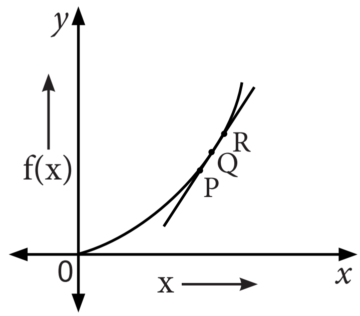
The word 'limit' is the best descriptor of the value of the function, when there is the smallest change in the value of the variable, say x. Formally, the 'limit of the function f as x goes to c is t' can also be rephrased as 'As x approaches c, the value of the function f gets arbitrarily close to t'.

In real life, when a chemical reaction between two chemicals takes place, a new compound is formed as time passes. So here, the new compound is the limit of a function as the time approaches infinity.

Similarly, tossing a coin gives a head or a tail. To know the probability of the outcome, we may flip a coin many times, making repeated trials. Here, as time approaches a more considerable period, the number of heads becomes equal to the number of tails in general. So, the limit of tossing a coin is the probability of getting an equal number of heads or tails as time approaches infinity.

**Continuity solves the conceptual challenge of instantaneous values**

The entire concept of limit hinges on how effective is the chosen infinitesimal value in detecting the rate of change or the slope of a function at the chosen instant. For example, while finding the best approximate value of the slope of $f(x) = x^2$ at the point Q (2, 4), the points P and R are moved as close as possible to Q. The reliability of the computed rate of change at an instant, is measured in terms of how consistent is the value of the rate, i.e., how close are the slopes of the tangent at P and tangent at R.




*The slope of the function f(x) at point Q*

One of the more obvious ways and means of seeking consistency is to look for the values of the rate at instants very close to the chosen instant. It is easily appreciable that the consistency of the rate of change of a function would be considered higher if at the two instants around the chosen instant – before and after – the computed values of the rate (before and after) are the same as the rate value at the chosen instant.

Thus, the infinitesimal value should be such that it can detect sharp variations in the values of the rate, closest to the point of interest (for finding an instantaneous value). The chances of capturing any sharp variation increase as the infinitesimal becomes smaller (and comes closest to the value of the instant). Any detected sharp variation declares a lack of continuity at that point.

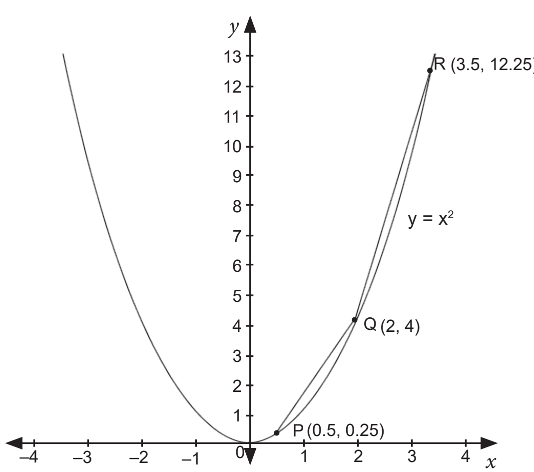
Continuity is an important consideration for finding derivatives, it helps to know if a function may not have a derivative at an instant (non-continuous functions do not have derivatives, we can know this without having to attempt a computation of the derivative), but it is not a necessary condition for computing the non-derivative value of a function over a range.

A digital recording of a song is an example of a continuous function. The digital recorder records tiny bits of sounds several times a second which may provide sufficient data for a computer to replicate the singer's overall performance while singing.
The growth of nails in human hands and feet is another example of continuous function. The nails grow at an average rate of 3.47 millimetres (mm) per month, or about a tenth of a millimetre per day. It grows and slides along the nail bed (the flat surface under the nails), giving strength to the nail. This process continues until the death of a human being. However, some factors that affect this continuous growth of function are age, location, season, hormones, health, etc.

**Ascertaining continuity at a point**

Let us graphically see how important is the choice of infinitesimal in appreciating the concept of continuity of a function (as evident from its curve) at a point. For a function to be continuous at a point, it is obvious that the slope of the tangents on the points just before and after the point of interest is nearly the same.

Take any random point on the curve $y = x^2$, say, P (0.5, 0.25) and R (3.5, 12.25) below and above the point of interest (point Q), and join points PQ and QR.

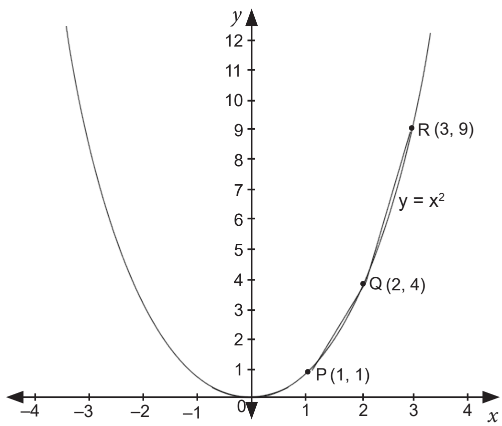



*Graph of $y = x^2$ with points P (0.5, 0.25) and R (3.5, 12.25)*

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0.25}{2 - 0.5} = \frac{3.75}{1.5} = 2.5$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12.25 - 4}{3.5 - 2} = \frac{8.25}{1.5} = 5.5$$

Now we move points P and R closer to point Q to calculate a better approximate value of the slope of the curve at Q. Let us choose P (1, 1) and R (3, 9), which are closer to Q (2, 4).

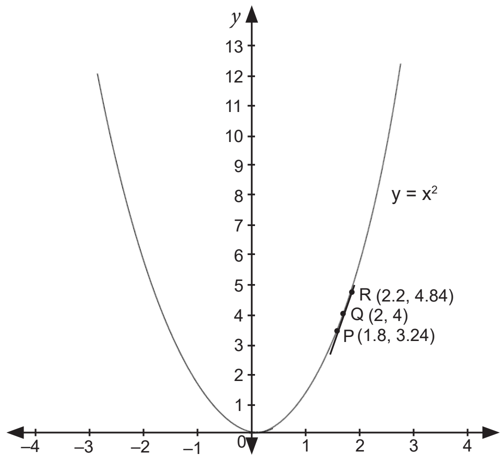



*Graph of $y = x^2$ with points P (1, 1) and R (3, 9)*

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 4}{3 - 2} = 5$$

Similarly, take another set of points, say, P (1.8, 3.24) and R (2.2, 4.84), which are further closer to point Q (2, 4).

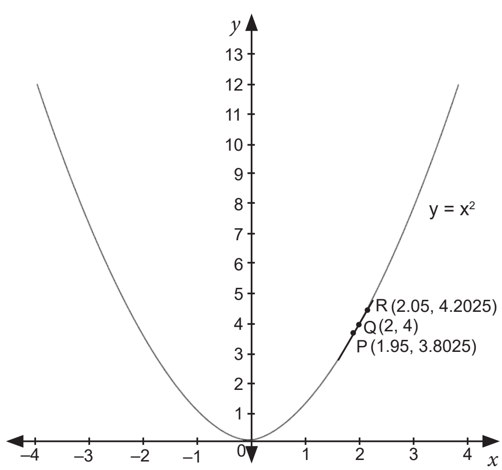



*Graph of $y = x^2$ with points P (1.8, 3.24) and R (2.2, 4.84)*

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3.24}{2 - 1.8} = \frac{0.76}{0.2} = 3.8$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.84 - 4}{2.2 - 2} = \frac{0.84}{0.2} = 4.2$$

Now moving very close to Q, take points P (1.95, 3.8025) and R (2.05, 4.2025).




*Graph of $y = x^2$ with points P (1.95, 3.8025) and R (2.05, 4.2025)*

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3.8025}{2 - 1.95} = \frac{0.1975}{0.05} = 3.95$$

$$\text{Slope of QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.2025 - 4}{2.05 - 2} = \frac{0.2025}{0.05} = 4.05$$

As points P and R get closer and closer to the chosen point, Q, the lines PQ and QR will coincide, eventually forming a tangent at point Q. The closer the points P and R are near the chosen point of interest Q, the closer are the values of the slopes at these points, suggesting a continuous function.

**Other examples of continuous change beyond motion**

*Average or indicative rate of reaction* is an important characteristic of chemical reactions. It is an essential parameter in large-scale manufacturing of chemicals, drugs, and household chemicals. For instance, knowing the rate at which products are being made and the bottlenecks (which may mostly be due to the lower-than-anticipated speed of reactions) production process can be fine-tuned.

For a chemical reaction, the change in concentration of reactants or product per unit time (such as second, minute, or hour) over a given period of time is called the average rate of reaction. And in a reaction, the rate of change of concentration of the reactants or products at a particular instant of time is the instantaneous rate of that reaction at a specific instant of time.

An interesting feature of the rate of reaction is that it continuously changes during every reaction – it depends upon the residual concentration of the reactants (which decreases with each passing instant of the reaction). A reaction never proceeds at the average rate of reaction. To really understand a chemical reaction, we need to go beyond the average rate of reaction.

*Stock market* intraday-trading involves traders buying and selling financial instruments based on fluctuating prices on the same day.

The trader makes a profit or loss based on the instantaneous stock price. This signifies a continuous process.

Whereas, in long-term investment, we look at the average of the stock prices and then invest in those stocks, which gives a good average return

$$\text{Average share price} = \frac{\text{Total cost of the shares purchased}}{\text{Number of share}}$$

*Simple and compound interest* rates are a matter of common knowledge (if not understanding) and experience that we could easily extend to broaden the appreciation of the difference between the basic idea/concept of average and instantaneous values of quantities that frequently or continuously change in time or space (i.e., change with change in position). It may also be added that when we talk about the 'real world change,' it implies that we cannot completely predict the change.

Simple interest rates are kind of 'average' of interest rates. The same flat interest rate will be applied for computing the interest amount on a principal over a period of time. The interest is assumed to be the same amount every day in that period. On the other hand, the compound interest rate resembles the idea of instantaneous interest amount – which varies by the day, week, month, or whatever period of compounding – over the deposit period.

This is to bring out that the instantaneous value of the interest amounts would behave differently under simple and compound interest situations.

*This chapter is excerpted from `Calculus for Professionals,' Volume I, co-authored by Sandeep Srivastava and Dr Garima V Arora.*

## Notes on slope, continuous quantities and differential equations

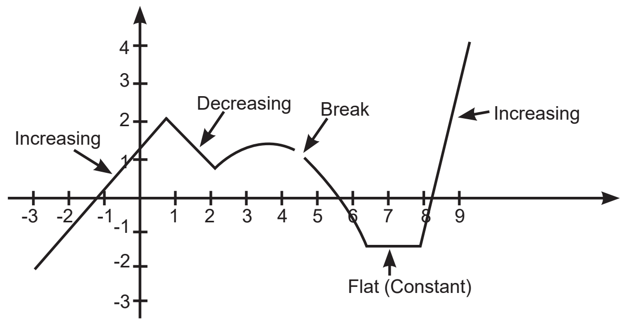
### *Note 1 Slope is important*

The slope gives the following information about a function:

- Steepness of a function.
- Direction of change of the curved line.
- The slope describes the degree of sensitivity of the dependent variable (y) on the independent variable (x) for a function y = f(x), i.e., the quantum of change in y due to infinitesimal change in the x. For example, a slope of 4 at a point means the y-axis will grow 4 times the (small) change in the x-axis.
- The slope of the function helps us compare any set of functions to know if they are parallel, perpendicular, or converging and the rate of convergence.
- The maximum and minimum value of a function – local (within a limited range of the variables) or global (over the entire range of values).
- The slope can be used to find whether the function increases or decreases after the location of the point of maxima or the minima. Indeed, the most important characteristic of non-linear functions is their slope.

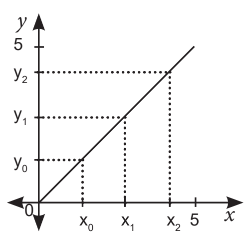
### *Note 2 Nature of the graphs and slopes of the functions*

Calculus is constructed on functions; a familiarity with the nature of the graphs of functions is required for understanding calculus. The graphs of the functions may be increasing or decreasing. They may be flat or may not even exist. They may occur with breaks or no breaks. This information about the nature of the graph speaks about the slope of the curve at various points.

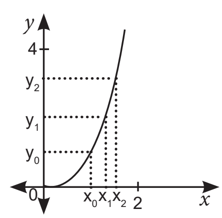



*Nature of the graphs - Increasing, decreasing, break, flat*

The instantaneous value of (constantly) changing quantities can only be found through the knowledge of the rate of change at all instants. The slope is the way to find the rate of change of the function, i.e., it describes how rapidly the outcome of a function changes with a unit change in its input(s) at various points in its domain. It tells about the steepness and direction of the lines and curves. Graphically, the rate of change is the slope.



*Graph of uniform rate of change*



*Graph of non-uniform rate of change*

Mathematically, the rate of change is the ratio of the change in the value of the function y, or f(x) due to a corresponding change in the value of x.

$$\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{(y_1 - y_0)}{(x_1 - x_0)}$$

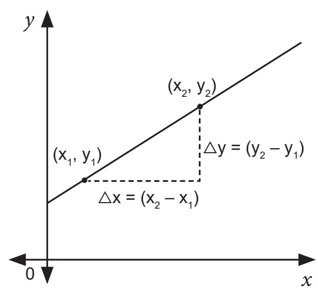
**Constant slope of a linear function**

It is easy to find the rate of change (or slope) of linear function, where the rate of change is constant.

Let $(x_1, y_1)$ be the point where the slope of the line is to be determined. Let us take any random point $(x_2, y_2)$ on the same line.

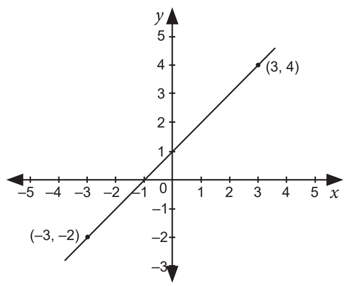
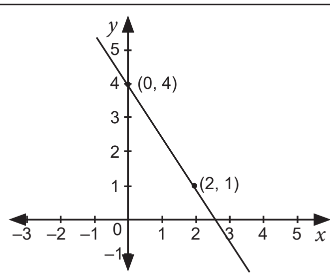
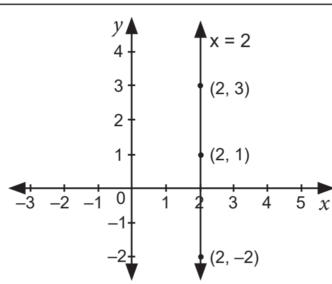
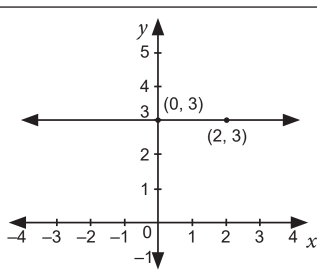
Then, the slope of the straight line is given by

$$\text{Slope, } m = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



**Graphical representation of a straight line**

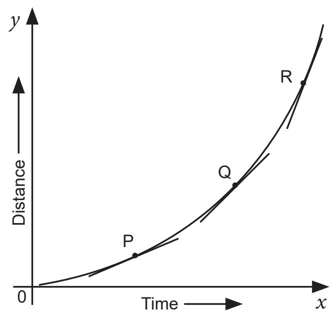
Now, let us consider the four cases mentioned below:

| Case | Graph | Slope |
|---|---|---|
| ***Case I*** Consider a line joining the points (3, 4) and (−3, −2). |  | An increasing function describes a positive slope. $m = \frac{4-(-2)}{3-(-3)} = 1$ |
| ***Case II*** Consider a line joining the points (0, 4) and (2, 1). |  | A decreasing function describes a negative slope. $m = \frac{1-4}{2-0} = \frac{-3}{2}$ |
| ***Case III*** Consider a vertical line x = 2. |  | A vertical line describes an undefined slope. $m = \frac{3-1}{2-2} = \frac{2}{0} = \infty$ |
| ***Case IV*** Consider a line joining the points (0, 3) and (2, 3). |  | A horizontal line describes zero slope. $m = \frac{3-3}{2-0} = 0$ |

From above inferences, we can say that the slope (or rate of change) of a linear function is always constant.

**Varying slopes of a curvilinear function**

The slope (or the rate of change) of the curvilinear function is not constant and keeps changing along the points on the curve. A tangent at a point on a curve is a straight line that best approximates the slope of the curve near that point.




*Tangents at various points on the curve*

**Tangent best approximates the slope of curve**

Let us see for ourselves how the slope of a curve at a point is best approximated by the tangent at that point (there can be only one tangent at a point on a curve).

We start with an ellipse with tangent PR at the point Q. From the image (i), it is not evident that the slope of PR is the same as the slope of the ellipse at point Q.

On enlarging the image and reducing the tangent PR (image ii) and we still cannot see the relationship between the slopes of PR and the ellipse at Q.

We further reduce the size of the tangent, and enlarge the diagram to view the relationship between the tangent and the curve of the ellipse.

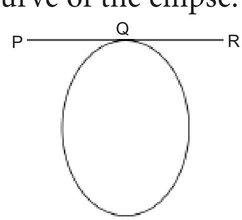



*Image (i)*

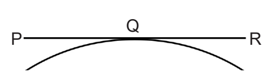



*Image (ii)*

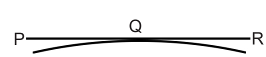
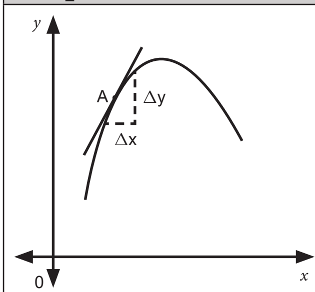
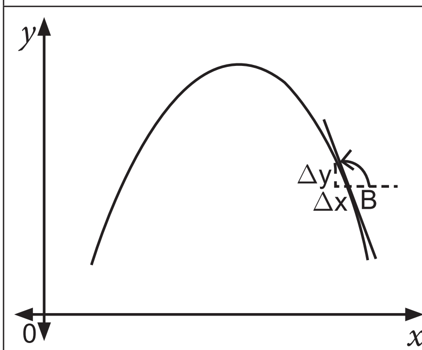


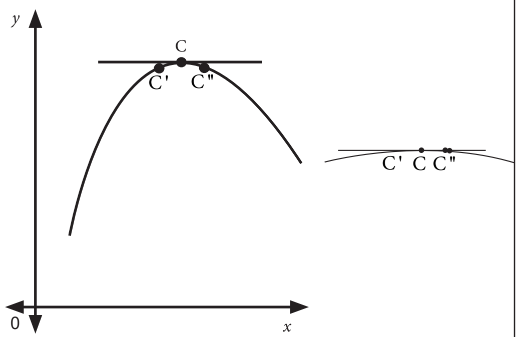
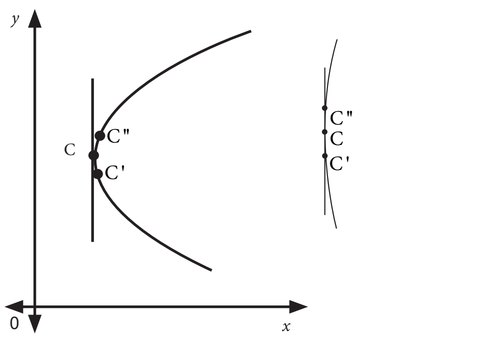

*Image (iii)*

Interestingly, as we keep enlarging the image and reducing the size of the tangent to know the preciseness of the slope of the curve at point Q, we will notice that a very small area of the curve (or a point) coincides with the tangent.


As we have already discussed, it is easy to calculate the slope of a straight line. To find the slope of a curve at a specific point, we find the slope of the tangent line at that specific point, it provides the best approximation. In the diagram, the slope of the tangent PR is the best approximate value of the slope of the elliptical curve at Q, as the tangent PR coincides with the curve at Q.

**Characterisations of slope for a curve**

| Graph | Slope |
|---|---|
|   | • The tangent to the curve at the point A is tending upwards when moving from left to right, which is a property of increasing functions – positive slope.<br>• The tangent at A makes an acute angle with the horizontal. |
|   | • The tangent to the curve at the point B is tending downwards when moving from left to right, which is a property of decreasing functions – Negative slope.<br>• The tangent at B makes an obtuse angle with the horizontal. |

| | |
|---|---|
|   | • As we move from C' to C and C to C" with an increase in x, the value of the function remains the same.<br>• The rate of change in both cases being 0. Thus, a horizontal tangent has slope 0. |
|   | • As we move from C' to C and C to C", the value of the function increases for the same value of x.<br>• The slope is not defined for a vertical tangent on the curve at a point. |

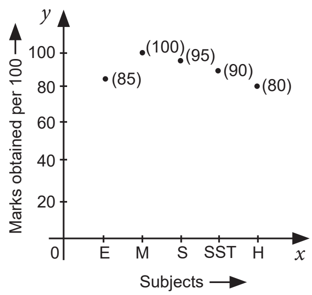
***Note 3 The nature of input and output***

**Nature of input**

The nature of input effects the nature of output. Let us explore how input values for a function may be different.

There is a world of things and situations that are quantified through the act of counting, and a world that is quantified through the act of measurement. The marks obtained by a student in an exam is quantified by the evaluator by first counting the marks obtained in individual questions and then adding them all up, whereas the weight of students in a class is quantified by measuring the weight of individual student on a weighing scale. Recall, functions take some kind of quantities as inputs and produce some other, or similar kind of quantities as output, which could be obtained by counting, or by measurement.

The nature of quantities obtained by counting is what we call as discrete. For example, marks obtained by a grade X student in all subjects in a school have discrete values, graphically seen as follows.




*Graphical representation of discrete quantities*

Discrete does not necessarily mean integer value, it only means that the possible values are definite and known; for example, the marks in a subject could well be 85.5, 85.25 or 85.75, but hardly 85.55.

The nature of quantities obtained by measurement/computation is typically what we call continuous, expressed using real numbers. Continuous quantities can take any value in an interval. For example, the aggregate percentage value is technically discrete because aggregate percentages are calculated by dividing the marks obtained by the total marks, which itself are discrete. However, percentage data is often treated as continuous for the reason that it can take any value from 0 to 100%.

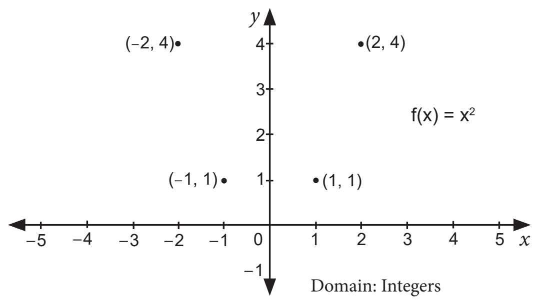
Thus, input values could be discrete or continuous, depending on how they originate – out of counting, measurement, or arithmetical computing.

**Discrete or continuous function – The nature of output**

The nature of output could also be discrete or continuous. The effect of the function on the nature of outcome is obvious; for example, $f(x) = x^2$ accepts inputs as negative and positive numbers but the outputs are only positive.

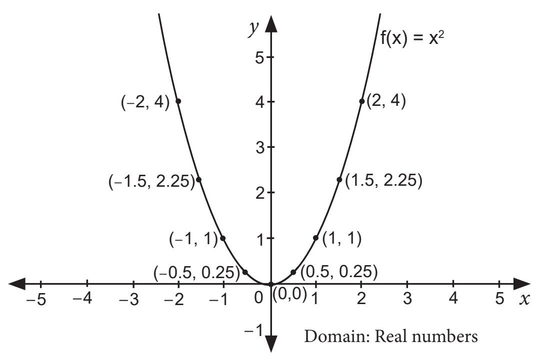
Let us take a function f(x) = $x^2$ and take different values to know how this function behaves.

When input values are taken as discrete, say, –2, –1, 0, 1, ..., the nature of output is also discrete and it is 4, 1, 0, 1, …, corresponding to the input values in the given function.




*Graphical representation of function for discrete input values*

On the other hand, when the domain is an interval (say [–2, 2]), the output or the range is also an interval (here [0, 4]) and takes every value between –2 and 2, and hence continuous.




*Graphical representation of function for continuous input values*

Thus, for a different domain of input values, the same function f(x) = $x^2$ will have a different range of the output values.

The nature of output depends on the nature of input and the nature of the function (i.e., the kind of 'processing').

The idea of a function being continuous is especially important in calculus.

### *Note 4 Mathematical expressions using derivative (Differential equations)*

Recall that algebraic expressions are combinations of constants and variables which are put together using mathematical symbols and algebraic equations are expressions that are set equal to zero. It is indeed fascinating to consider that equations can incorporate changing conditions and the rate of change through the use of derivatives and rates of change. Such equations are common, they are mathematical tools used for modelling and analysing everyday situations and scientific conditions that involve change and relationships between variables. These 'derivative equations' are mathematical termed as 'differential equations'.

Do not be startled if we say that the idea and application of 'differential equations' is a primitive biological, animal instinct and ability. We are adept at using differential equations intuitively.

For starters, in a race, gauging and reacting in the heat of the moment to the increasing or decreasing distances between the runners as things change dynamically involves using the brain's raw/god-gifted logical abilities and spontaneous calculations. What we do intuitively, formal mathematical modelling of the situation would involve calculus! The conscious and engaging assessment and extra push mid-air when jumping across a ditch ensures that the jump is successful. Again, a mathematical modelling of decision making in the situation would be based on the derivative (rate of jump) and anti-derivative (the extent of jump).

Wherever there are changing quantities in the 'equation' of thing, the situation is mathematically expressed as differential equations. These equations can be used to configure everyday life to rocket science. The laws of nature and dynamism in science and maths can easily be explained through differential equations. Think of the way we go across a busy road – constantly juggling with the estimated speed of the vehicle (rate of approach), the closing distance between the fast-approaching vehicles and the person crossing the road, the distance left to cross the road, speed of the person crossing the road, as well as the obstructions on the way (the other people crossing the road from the opposite direction, for instance); it is a fairly complex situation of changing dimensions, but most of us have gotten it right every time.

Some examples of differential equations in real life:

- Any change in human body temperature is a response to changing conditions outside and within the body, such as ambient temperature, the type of food eaten, the type of clothes worn, the type of activity performed at that particular time (for example, exercising would increase the heart pumping and blood circulation rate, thereby increasing the temperature), and more. So, 'in the equation', the temperature of a human body responds to various changing conditions underlying it.
- For calculating the time required to drain a tank full of fluid, differential equation comes into play. Draining time depends on various factors like the volume of fluid, the air pressure, the height of the tank, the density of the fluid, the rate of flow, the size of a draining hole, and more. Any change in the above factors may impact the time taken to drain the tank. For example, if water and petroleum are put in two similar tanks, the time taken to drain them would differ due to the difference in their density. Similarly, if the size of the draining hole is small, it would impact the time taken to drain the fluid. 'In equation terms', factors like the size of the hole and height of a tank are constant terms for a specific tank for all fluids, while density, the volume of fluid, and air pressure are differentiated with time to estimate the time taken to drain the fluid.
- The value of the National income of a country is dependent on various factors like general price level, aggregate demand, aggregate supply, compensation to employees, saving rate, government policy, and more. These factors depend on the inflation rate, total production of goods and services, wage rate, marginal propensity to consume, etc. All the factors are

interlinked and dynamic in nature. For example, the general price level of a country increases due to an increase in the inflation rate, which may change due to government policy or a change in supply. 'In equation term', compensation of employees and saving rate are constant terms for a period while general price level, aggregate demand, and supply are differentiated with time to know the value of national income.

- The differential equation is used in a video game to determine the rate of motion of an object. For example, consider the static force diagram for a ball rolling down a ramp. Knowing the time duration in which it will roll down depends on various factors like gravity vector, mass of the ball, and the angle of the ramp (its normal vector).These factors are further dependent upon the net force applied on the ball and the acceleration of the ball in that frame. So, 'in equation terms', the mass of the ball and gravity vector is considered constant, while other factors are variable and differentiated with respect to time.

An interesting aspect of differential equations is that unlike algebraic equations and the math we know, the 'solution' of differential equations is not a quantity (or a set of quantities) but another function. Such a solution might be expected because when we deal with derivatives, we essentially deal with functions. It means that differential equations give us a 'modelled' behaviour of things, not a particular instance of behaviour. And there is often a set of solutions for a given differential equation.

A few famous equations in physics which depict the rate of change are:

Force = Mass × Rate of change of velocity
Power = Voltage × Rate of change of charge
Momentum = Mass × Rate of change of distance
The conceptual exploration of derivative (and the related idea of 'anti-derivative and derivative equation') concludes here.

# It's Your Convenience World, finally

*This is for real, for once*

> *"Welcome to the New World Order – the IYCWorld (it's-your-convenience-world). The power to choose yourself and engage in your chosen socio-economic proclivities, around the world, at the click of a mouse is a force that will transform your life like nothing before. And your choice will be your limit in the skyless e-universe. IYCWorld is only as good as you demand it to be ... try as they might, the local socio-economic dimensions cannot stop your will and convenience to rule."*
>
> – *Sandeep Srivastava, 2001*

This is an excerpt from the 2001 book 'Embracing the Net,' published by FT.com (Pearson, UK,) co-authored by Soumitra Dutta, currently Dean Said Business School, University of Oxford, and Sandeep Srivastava. The extract was the stated overarching vision and strategic direction for the digital economy in the Third Industrial Revolution (3IR), post the 2000 dot-com bust.

Twenty-three years later, and a decade in the Fourth Industrial Revolution, 4IR, the vision and strategic prescription is just as valid and robust, literally nothing even to be tweaked. Indeed, we have come a long way in the right direction, the emergence of the 4IR as the hard-infrastructure for Society 5.0 is just the needed 'physical enabler'. However, we are far too away from transformations on the ground.

It is not too hard to locate why we are still as long a way from Society 5.0, in Japan as much as in every nation of the world – the complementing soft-infrastructure of the 4IR is missing in action. A vast majority of the educated adults of our times cannot harness 4IR, and it is turning for the worse. Education – the technology of raising best-potential adults out of every child – has turned out to be the intractable complication for all of humankind.

However, mathematisation of mathematics education is a masterstroke for ushering in an educational renaissance. For, learning mathematics is peerlessly personalisable and most objectively evaluable. Besides, mathematics is the easiest domain of knowledge to learn; every one of us is born with all the mathematical logic that is there is to be discovered, waiting to decipher the order in the nature. Success in mathematics education is the only first step way to kick start the larger educational reformation, and we can go on.

Above all, 'Cent Percent Mathematics' is no more than 50 hours of conversation for the entire K-12 curricula. And language is no barrier. Cent Percent Mathematics is all in public domain, the evidence of which are the two case studies. Mathematics-led educational revolution is real now, ready to more than complement 4IR and set off a virtuous cycle of growing economic dignity to every one of us.

*Be ready to play your part, in mathematising your own mathematics, and experience economic miracles for self, family, and community.*

*It is essential to realize that science does not offer a complete knowledge of the mind, although we do experience its mystery and enormous energy. However, it is clear that energy is a vital prerequisite for performing mechanical work.*

*The mental processes of the mind are essential for performing creative, innovative work, the difference lies in how people utilize the power of their mind. Everyone knows the saying that "an empty mind is Devil's workshop," so without a meaningful purpose, people might spend their mental energies on destructive work. On the other hand, mental energy can be utilized for creative or innovative work as well as improving quality of life.*

*A mathematised mind is highly predisposed towards seeking and seeing order in all things around.*

*– Ramjee Prasad, 2012*

4IR may be our only window to secure virtuously sustainable social ecology, as sci-tech intensity gathers unprecedented pace and sweep. To top it all, big-data-tech is peerlessly democratising research and innovation. But every technology is 100% mathematised knowledge in action.

Mathematisation of thinking is building natural-language-like competence in expressing real, or imagined relationship of quantities. On the ground, it boils down to intensively exploring the conceptual foundations in place of 'rigourous, calibrated mathematics' to actualise mathematics as a language.

In 'Cent Percent Mathematics' is humanity's first tool to cultivate and harness the hands and minds of all 8 bn us, ensure economic miracle, true democracy, and greener earth hereafter.


Prof. Ramjee Prasad is President, CTIF Global Capsule, Professor Emeritus, Department of Business Development and Technology, Aarhus University, Herning, Denmark.


He is variously recognised by governments and industry for internationalization of top-class telecommunication research and education.

He has published more than 50 books, 1000+ journal & conference papers, over 15 patents, and over 150 Ph.D. Graduates. He is the recipient of Knight ("Ridder") of the order of Dannebrog and 'Pravasi Bhartiya Samman' (Overseas Indian Award.)

Sandeep Srivastava, educated at Jadavpur University, IIT Delhi and INSEAD, is the first Indian and among the pioneering few, globally, to author three books about the Internet in 2000-2, published in UK, USA, and India. He is comprehensively committed to K-12 reform. He is all set to empower parents to guarantee overall development of every child, across the world. He has authored 'Cent Percent Mathematics' to power a peerless human revolution wherein every adult harnesses AI and 4IR for dignity.

*For more information scan the QR code*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